Characteristic classes on complex manifolds and Chernnumber inequalities on compact Kähler surfaces

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Introduction

The most important topological invariant is the *Euler characteristic* denoted by χ. The very first definition of χ is for a finite simplicial complex M:

$$\chi(M) = \sum_{i=0}^{\infty}(-1)^i h^i(M, \mathbb{Z}),$$

where $h^i(M, \mathbb{Z})$ is the dimension of the i-th singular homology $H_i(M, \mathbb{Z})$, which is the number of basis of i-complexes in M. For M a compact oriented differentiable manifold, which is in particular a finite simplicial complex, the Euler characteristic of M is also defined. Hopf generalized a method of calculating the Euler characteristic of a compact differentiable manifold M by counting the index of a generic vector field over M. This is known as the Poincaré-Hopf theorem. It is remarkable since it provides a differential geometric way to calculate a topological invariant.

This way is further generalized by Whitney. He is able to use k generic vector fields instead of only one. In the process, the *Stiefel-Whitney class* $w^{n-k+1} \in H^{n-k+1}(M)$ of tangent bundle of M, which is dual to the homology class represented by the zero set of the k vector fields on M. This is a proper generalization since when $k = 1$, the pairing of w^n with the fundamental homology class $[M] \in H_n(M)$ again gives the Euler characteristic. The notion of Stiefel-Whitney classes survives if the tangent bundle is in general a vector bundle. Following this, there are *Pontrjagin classes* for vector bundles over real $4n$-dimensional manifolds and the *Chern classes* for complex vector bundles over complex manifolds. S. S. Chern pointed out that the characteristic classes are topological invariants of vector bundles which measure the difference of a vector bundle from a product structure. He also gives a curvature form representation of the Chern classes.

From this, it is natural to consider problems like whether certain condition on curvature of a vector bundle will cause any relation between Chern numbers. We will study four Chern-number inequalities on complex surfaces: The first one is $c_1^2 \leq 3c_2$ on Kähler surfaces with no restriction on curvature. If the surface is Kähler-Einstein with negative Einstein constant, then the equality holds if and only if the surface is biholomorphic to a ball. The second one is $c_1^2 \geq c_2$ on Kähler surfaces with nonpositive bisectional curvature. The third inequality is $c_1^2 \geq 2c_2$ on Kähler surfaces with nonpositive sectional curvature. The fourth one $(r-1)c_1^2(E) \leq 2rc_2(E)$ is on a rank r stable bundle E over a compact Kähler manifold by using a Hermitian-Einstein metric, which can be shown to exist on the stable bundle. The resulting inequality for rank 2 bundle is $c_1^2(E) \leq 4c_2(E)$.

As an analogue of the Euler characteristic for compact differentiable manifolds, there is the *Euler-Poincaré characteristic* for a complex vector bundle E over a compact complex manifold M^n defined by

$$\chi(E) = \sum_{i=1}^{n}(-1)^i h^i(M, E),$$

where $h^i(M, E)$ is the dimension of the i-th cohomology group of the vector bundle E. In the manifold case, the Euler characteristic can be calculated by the characteristic classes. In the vector bundle case, Todd in 1937 gave an independent result of calculating the Euler-Poincaré characteristics by a Todd polynomial with the Todd classes as indeterminates, but the proof was incomplete. Later in 1953, Nakano gave the equivalence of the Todd class and the Chern class. With this motivation, Hirzebruch in [Hir] gave a complete proof of the expression of the Euler-Poincaré characteristic in terms of Chern classes in a Todd polynomial, which is known as the *Riemann-Roch theorem of Hirzebruch.* We will also present his proof in this paper. As an application, the Riemann-Roch theorem will be used in the last chapter to calculate the Euler-Poincaré characteristic of certain vector bundles.

The last chapter of this paper is devoted to studying a slightly different topic, namely the deformation theory of complex structures. The theory was invented by Kodaira and Spencer. To get a concrete view of deformation theory, we first study the complex analytic family of compact complex manifolds parameterized by a complex manifold and the Kodaira-Spencer map. Then we study the moduli space of complex structures of a compact complex manifold, in particular, Kuranishi's bound of freedom of deformation. As an analogue of deformation of compact complex manifolds, there is deformation of vector bundles, which are dealt with as complex manifolds with locally product structures. We will concentrate on the deformation of vector bundle V with fixed compact complex 2-dimensional base manifold M but not obtained by tensoring with a family of not all trivial line bundles alone. In the 2-dimensional case, we may represent Kuranishi's lower bound of freedom of deformation $h^1(M, End_0V) - h^2(M, End_0V)$ by $-\chi(M, End_0V) + h^0(M, End_0V)$ which is greater than or equal to $-\chi(M, End_0V)$. This Euler-Poincaré characteristic can be represented in Chern classes of M and Chern characters of End_0V by using the Riemann-Roch theorem. When $V = TM$, the bound is given by polynomial of Chern classes of M and we can relate the bound to the inequality $c_1^2 \leq 3c_2$ of Chern numbers.

CHAPTER 1

Differentiable Manifolds and Euler Characteristics

We will first give the definition of Euler characteristics defined by singular (co)homology from [Greenberg] and then give the basic notions related to differentiable manifolds, including Riemannian metric, connection and curvature of Riemannian manifolds, Hermitian metric and Kähler metric in which the main references are [do Carmo] and [Griffiths]. For later use, this chapter also includes the definition of Čech cohomology and de Rham cohomology of manifolds. The references are [Hir] and [Madsen]. Finally, we give the Hopf-Poincaré theorem, which gives the Euler characteristic of a compact manifold in terms of index of a vector field over the manifold. This result serves as motivations of later works of Todd, Nakano and Hirzebruch. The reference is [Madsen] Chapter 11, 12.

1. Singular Homology and Singular Cohomology

This section gives the preliminary in algebraic topology and gives the definition of Euler characteristic. We will omit most of the proofs that can be found in [Greenberg].

1.1. Singular Homology.

1.1.1. *Singular Chain Group.* Let X be a set and if for each point $x \in X$ there corresponds a nonempty collection of subsets of X, which we call them *neighborhoods of* x satisfying

(i) x lies in each of its neighborhoods;
(ii) The intersection of two neighborhoods of x is again a neighborhoods of x;
(iii) If U is a neighborhood of x and V be a subset of X which contains U, then V itself is a neighborhood of x;
(iv) If U is a neighborhood of x, then its interior U° defined by the set of elements in U with U as neighborhoods, is also a neighborhood of x.

Then the whole structure is called a *topological space*. The assignment of a collection of neighborhoods of a set X satisfying the four conditions are called a *topology* on the set X. *Morphisms* between topological spaces are continuous maps. A property of a topological space which is preserved under continuous maps is called a *topological invariant*. To give the very first definition of the *Euler characteristics* of topological spaces, we will give the definition of singular homology first. The set

$$\Delta^q = \{(t_0, t_1 \ldots, t_q) \in \mathbb{R}^{q+1} : t_0 + t_1 + \cdots + t_q = 1, t_i \geq 0, i = 0, 1, \ldots, n\}$$

is called a *standard q-simplex.* A continuous map σ, from a standard q-simplex Δ^q to a topological space X is called a *singular q-simplex in X*. Given a topological space X and a fixed nonnegative integer q, the singular q-simplexes can be added and subtracted formally. Let Ω be a commutative ring with unit, define formal linear combinations $\sum_\sigma a_\sigma \sigma$, where the summation runs through all the singular q-simplexes of X and each a_σ is an element in Ω. The formal sum is then called a *singular q-chains in X*.

Definition 1.1 (Singular Chain Group). The set of all the formal sums defined above,

$$C_q(X, \Omega) = \left\{ \sum_\sigma a_\sigma \sigma, a_\sigma \in \Omega \right\}$$

is called the *q-th singular chain group of X with coefficients in* Ω. If q is negative, then the group is defined to be the trivial group.

1.1.2. *Singular Homology Group.* For q a nonnegative integer, the *i-th face* $(0 \leq i \leq q)$ of a singular q-simplex $\sigma : \Delta^q \longrightarrow X$ is the singular $(q-1)$-simplex

$$\sigma^{(i)} = \sigma \circ \phi_i : \Delta^{q-1} \longrightarrow X,$$

where the map ϕ_i is a linear map from Δ^{q-1} to Δ^q defined by

$$\phi_i(t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_q) = (t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_q).$$

Definition 1.2 (Boundary Operator). Define the *boundary operator* of a singular q-simplex σ of a topological space X to be the singular $(q-1)$-chain

$$\partial\sigma = \sum_{i=0}^{q} (-1)^i \sigma^{(i)}.$$

Proposition 1.1. $\partial\partial = 0$.

Proof. It suffices to show $\partial\partial\sigma = 0$ for all σ being singular q-simplex, since all the q-singular chains are linear combinations of q-simplexes. The proof uses the following identity: $\phi_i \circ \phi_j = \phi_j \circ \phi_{i-1}$ where $0 \leq i < j \leq q$, which can be computed directly. For any singular q-simplex σ,

$$\begin{aligned}
\partial\partial\sigma &= \sum_{i=0}^{q}(-1)^i\partial(\sigma\circ\phi_i) \\
&= \sum_{i=0}^{q}(-1)^i\sum_{j=0}^{q-1}\sigma\circ\phi_i\circ\phi_j \\
&= \sum_{i=1}^{q}\sum_{j<i}(-1)^{i+j}\sigma\circ\phi_i\circ\phi_j + \sum_{j=0}^{q-1}\sum_{i\leq j}(-1)^{i+j}\sigma\circ\phi_i\circ\phi_j \\
&= \sum_{i=1}^{q}\sum_{j<i}(-1)^{i+j}\sigma\circ\phi_j\circ\phi_{i-1} + \sum_{j=0}^{q-1}\sum_{i\leq j}(-1)^{i+j}\sigma\circ\phi_i\circ\phi_j \\
&= \sum_{j'=0}^{q-1}\sum_{i'<j'+1}(-1)^{i'+j'-1}\sigma\circ\phi_{i'}\circ\phi_{j'} \\
&\quad + \sum_{j=0}^{q-1}\sum_{i\leq j}(-1)^{i+j}\sigma\circ\phi_i\circ\phi_j = 0.
\end{aligned}$$

□

Let c be a singular q-chain. If $\partial c = 0$, then c is called a *singular q-cycle*. If there exists some singular $(q+1)$-chain c' such that $c = \partial c'$, then c is called a *singular q-boundary*. Denote

$$Z_q(X,\Omega) = \{q\text{-cycles in } X\},$$

$$B_q(X,\Omega) = \{q\text{-boundaries in } X\}.$$

By Proposition 1.1, $B_q(X,\Omega)\subset Z_q(X,\Omega)$, thus the following definition is well-defined.

Definition 1.3 (Singular Homology Group). The identification space

$$H_q(X,\Omega) = Z_q(X,\Omega)/B_q(X,\Omega)$$

is called the *q-th singular homology group of X with coefficients in Ω.*

1.1.3. *Euler Characteristics.* Let us consider the functorial properties of homology groups first. Let X and X' be two topological spaces and $f : X \longrightarrow X'$ be a continuous map. f induces a map

$$C_q(f) : C_q(X,\Omega) \longrightarrow C_q(X',\Omega)$$

by $C_q(f)(\sum_\sigma a_\sigma\sigma) = \sum_\sigma a_\sigma f\circ\sigma \in C_q(X',\Omega)$ for any $\sum_\sigma a_\sigma\sigma \in C_q(X,\Omega)$. Since for any two continuous maps $f, g : X \longrightarrow X'$ and id, the identity map, we obviously have

(i) $C_q(id) = id$;
(ii) $C_q(f\circ g) = C_q(f)\circ C_q(g)$

and therefore $C_q(f)$ defined above is a homomorphism. To further induce a map on homology group, we need:

PROPOSITION 1.2. $\partial \circ C_q(f) = C_{q-1}(f) \circ \partial$.

PROOF. Again it suffices to prove $\partial \circ C_q(f)(\sigma) = C_{q-1}(f) \circ \partial(\sigma)$ for any singular q-simplex σ of X. Compute

$$\begin{aligned} \partial \circ C_q(f)(\sigma) &= \partial(f \circ \sigma) \\ &= \sum_{i=0}^{q} (-1)^i (f \circ \sigma) \circ \phi_i \\ &= f\left(\sum_{i=0}^{q} (-1)^i \sigma \circ \phi\right) = C_{q-1}(f) \circ \partial\sigma. \end{aligned}$$

□

Proposition 1.2 guarantees $C_q(f)$ sending cycles to cycles and boundaries to boundaries, hence we can induce the homomorphism

$$H_q(f) : H_q(X, \Omega) \longrightarrow H_q(X', \Omega)$$

by $H_q(f)[\sigma] = [C_q(f)(\sigma)] \in H_q(X', \Omega)$ for any $[\sigma] \in H_q(X, \Omega)$. Suppose X and X' are homeomorphic as two topological spaces, then their homology group are also isomorphic. Therefore every homology group of a topological space is a topological invariant.

DEFINITION 1.4 (Euler Characteristic). Given a topological space X and the ring of integers $\mathbb{Z}$, the dimension of $H_q(X, \mathbb{Z})$ as a module over $\mathbb{Z}$ is the *q-th Betti number of* X, denoted as β_q. The *Euler characteristic* χ of X is defined as the alternating sum

$$\chi(X) = \sum_{q=0}^{\infty} (-1)^q \beta_q$$

whenever the sum is finite.

Since every homology group of X is a topological invariant, each Betti number is a topological invariant. Hence the Euler Characteristic of the topological space X is a topological invariant.

1.2. Singular Cohomology.

1.2.1. *Singular Cochain Group.* Let X be a topological space and Ω a commutative ring. Define a *q-singular cochain* to be an Ω-linear homomorphism

$$c : C_q(X) \longrightarrow \Omega.$$

DEFINITION 1.5 (Singular Cochain Group). Define the *group of singular q-cochains* $C^q(X, \Omega)$ to be the set of all such singular q cochains. Hence it is the dual of the group of singular chains, that is,

$$C^q(X, \Omega) = Hom(C_q(X), \Omega).$$

If q is negative, then the group is defined to be the trivial group.

PROPOSITION 1.3. *Denote the pairing* $[z, c]$ *of the homomorphism* c *and a* q*-chain* z *by the value of* c *on* z*, then for any* $z_1, z_2 \in C_q(X, \Omega)$*,* $\nu \in \Omega$ *and* $c, c_1, c_2 \in C^q(X, \Omega)$*, we have:*

(i) $[z_1 + z_2, c] = [z_1, c] + [z_2, c]$;
(ii) $[z, c_1 + c_2] = [z, c_1] + [z, c_2]$;
(iii) $[\nu z, c] = \nu[z, c] = [z, \nu c]$.

PROOF. Direct checking. □

1.2.2. *Singular Cohomology Group.*

DEFINITION 1.6 (Coboundary Operator). Define the *coboundary operator* by the homomorphism

$$\delta : C^q(X, \Omega) \longrightarrow C^{q+1}(X, \Omega)$$

by requiring $[\partial z, c] = [z, \delta c]$ for all $z \in C_{q+1}(X, \Omega)$ and $c \in C^q(X, \Omega)$. That is, δ is the adjoint of ∂ under the pairing $[\quad,\quad]$.

By $\partial\partial = 0$ in Proposition 1.1 and the adjoint property, we have $\delta\delta = 0$. Let z be a singular q-cochain. If $\delta z = 0$, then z is called a *singular* q*-cocycle*. If there exists some singular $(q-1)$-cochain z' such that $z = \delta z'$, then z is called a *singular* q*-coboundary*. Denote

$$Z^q(X, \Omega) = \{q\text{-cocycles in } X\};$$

$$B^q(X, \Omega) = \{q\text{-coboundaries in } X\}.$$

By $\delta\delta = 0$, $B^q(X, \Omega) \subset Z_q(X, \Omega)$, the following definition is well-defined.

DEFINITION 1.7 (Singular Cohomology Group). The quotient space

$$H^q(X, \Omega) = Z^q(X, \Omega)/B^q(X, \Omega)$$

is called the *q-th singular cohomology group of X with coefficients in Ω.*

1.2.3. *Poincaré Duality.* For any continuous map $f : X \longrightarrow X'$ between the topological spaces X and X', define the homomorphism

$$C^q(f) : C^q(X', \Omega) \longrightarrow C^q(X, \Omega)$$

by requiring $[z, C^q(f)c] = [C_q(f)z, c]$ for all $z \in C_q(X', \Omega)$ and $c \in C^q(X, \Omega)$. To further induce f onto cohomology groups, we need:

PROPOSITION 1.4. δ *as defined above, then for any continuous map* $f : X \longrightarrow X'$*,*

$$\delta C^q(f) = C^{q+1}(f)\delta.$$

PROOF. Direct Checking. □

If $f : X \longrightarrow X'$ is any continuous map between two topological spaces X and X', the following induced map is well defined:

$$H^q(f) : H^q(X', \Omega) \longrightarrow H^q(X, \Omega).$$

Hence cohomology groups are also topological invariants. We will state without proof the following important duality theorem:

THEOREM 1.1 (Poincaré Duality). *Let X be a n-dimensional compact oriented differentiable manifold, then*

$$H^q(X, \Omega) \cong H_{n-q}(X, \Omega).$$

PROOF. Refer to [Greenberg] p. 164. □

2. Manifolds and Metrics

This section gives the definition of differentiable manifolds, especially Riemannian manifolds, and complex manifolds, especially Hermitian manifolds and Kähler manifolds. The main references are [do Carmo] and [Griffiths].

2.1. Differentiable Manifolds.

DEFINITION 2.1 (Differentiable Manifold). Let M be a topological space, if there exists an open covering $\{U_i\}_{i \in I}$ of M such that for each U_i there exists a one-to-one map

$$x_i : U_i \longrightarrow \mathbb{R}^n$$

and for each pair i, j with $W = U_i \cap U_j$ nonempty, we have that $x_i(W)$ and $x_j(W)$ are open subsets in $\mathbb{R}^n$ and further the mappings $x_j \circ {x_i}^{-1}$ and $x_i \circ {x_j}^{-1}$ are differentiable maps. Then M is called a *differentiable manifold of dimension* n.

DEFINITION 2.2 (Tangent Space). Let M be a differentiable manifold. For any point $P \in M$, the vector space of all tangent vectors at the point P is called the *tangent space of M at P*, denoted as $T_P M$.

2.2. Riemannian Metrics, Connections and Curvatures.

2.2.1. *Riemannian Metrics.* Riemannian manifolds are fragments of Eucliean space with metrics piecing together nicely.

DEFINITION 2.3 (Riemannian Metric). If an n-dimensional differentiable manifold M is assigned a positive definite inner product $< \cdot, \cdot >_P$, operating on the tangent space at that point for each point $P \in M$, and if the assignment varies smoothly with respect to points in M, then M is called a *Riemannian manifold* and the set $\{< \cdot, \cdot >_P : P \in M\}$ is called a *Riemannian metric* of the Riemannian manifold M.

2.2.2. *Connections.* Let S be a surface in the Euclidean space $\mathbb{R}^3$. Considering a vector field V on the surface S, let $c : I \longrightarrow S$ be a curve on S. We can get the change of the vector field along the curve c. That is the projection of the directional derivative of V in the direction of the vector $c'(t)$ on to the tangent space of S at that point. This is called the *covariant derivative* of V in S along the curve $c(t)$.

Let S be a Riemannian manifold embedded in an ambient spaces, say M. Let $c(t)$ be a curve in S and V be a vector field on S, we can also get the projection of the derivatives of V along $c(t)$ onto the tangent space of S at that point. This is the covariant derivatives of the vector field V along a curve c in S. Now we fix a point P in S, and let v_P be a tangent vector of S. If we want to know the projection of directional derivative of any vector field V in the direction of v_P onto S, we can choose a curve $c(t)$ such that $c(0) = P$ and $c'(0) = v_P$ and define the projection of directional derivative of V in the direction of v_P to be the covariant derivative of V along the curve $c(t)$ at the point P. Actually, we can show that the projection of directional derivative of V is independent of the choice of curve $c(t)$. Further, if we can show that the projection of directional derivative is independent of the ambient space, then it is an intrinsic quantity of the manifold S and vector fields defined on it. In this way, we have actually defined an operator from two tangent vector fields to one tangent vector field. Details will be given. The operation of projection makes the defined operator an intrinsic property of the surface S and all vector fields on S. This operator is called *connection*.

Let M be a Riemannian manifold, $C^\infty(M, \mathbb{R})$ denote the set of all smooth vector fieldss on M and $A^p(M, \mathbb{R}), p \geq 1$ denote the set of all differentiable p-forms on M.

Definition 2.4 (Affine Connection of Riemannian manifolds). An *affine connection of M* is a map

$$\triangledown : C^\infty(M, \mathbb{R}) \longrightarrow A^1(M, \mathbb{R}),$$

satisfying the Leibniz's rule

$$\triangledown(f \cdot X) = df \otimes X + f \cdot \triangledown(X), \ X \in C^\infty(M, \mathbb{R}), f \in C^\infty(M, \mathbb{R}).$$

For any tangent vector field X and a tangent vector field Y, the contraction $\triangledown_Y(X) = (\triangledown X)(Y)$ is the covariant derivative of X in the direction of Y. In general there are infinitely many affine connections for a Riemannian manifold. However, there remains a unique one after two restrictions are imposed. An affine connection $\triangledown$ of a Riemannian manifold M with Riemannian metric $g = \{< \cdot, \cdot >_P\}_{P \in M}$ is said to be *torsion-free* if

$$\triangledown_X Y - \triangledown_Y X = [X, Y],$$

where $[X, Y] = XY - YX$ for all $X, Y \in C^\infty(M, \mathbb{R})$. $\triangledown$ is said to be *compatible with the metric* if

$$d < X, Y >_P = < \triangledown X, Y >_P + < X, \triangledown Y >_P$$

at any point $P \in M$ and for all $X, Y \in C^\infty(M, \mathbb{R})$.

Definition 2.5 (Riemannian Connection). An affine connection ∇ of a Riemannian manifold M is called a *Riemannian connection* if it is both torsion-free and compatible with the Riemannian metric.

Theorem 2.1 (Levi-Civita). *Given a Riemannian manifold M, there exists a unique Riemannian connection.*

Proof. Existence is shown by construction and uniqueness is shown by solving ordinary differential equations. The proof can be found in [do Carmo] p. 55. □

2.2.3. *Curvatures.* We have some intuitive idea of "curvature"(the quotation mark means this is not a mathematical object). It is interesting to induce a mathematical object to describe "curvature". Consider a point P in a surface S, and choose a small neighborhood A in S. Let B be an open set in the tangent plane of S at P which is correspond to A by some mapping. We may believe that the difference between the area of A and B can somehow describe the curvature of S at P. Further, we may assume the curvature of a flat plane be zero and the curvature of a sphere of radius r to be $\frac{1}{r}$. With these assumptions, we can get a definition of curvature. Let B be an open ball with radius r, the mapping between B and $A = expB$ is the exponential map, exp, and we have the definition of curvature as

$$K = \lim_{r \longrightarrow 0} 12 \frac{Area(A) - Area(B)}{r^2 Area(A)} \tag{1}$$

from [Helgason] p. 64. This is called the *Gaussian curvature* of a surface.

We may ask in the case of higher dimension, do we have a generalized mathematical object to describe "curvature"? The answer is positive. However it seems that we have at least two directions of generalization. The first direction may simply use formula (1) of Gauss but change the nominator to the volume of the manifold and the denominator to the volume of a same dimensional ball. The second direction may locally use a frame say $\{e_1, \ldots, e_n\}$ of the tangent bundle. Compute the Gaussian curvatures of the surfaces "generated" by any two of the basis (actually, the surface is the image of the plane generated by the two basis through the exponential map), which is called the *sectional curvatures* of the surfaces. Taking average of the sectional curvatures in a certain way, we can get the so called *scalar curvature*. This scalar curvature also agrees with our intuition of "curvature". The second generalization is adopted. What is remarkable is there is a simple expression of sectional curvature by commutativity of connections in different directions in terms of connections as mentioned in [Spivak].

Definition 2.6 (Curvature). Let M be a Riemannian manifold with Riemannian metric $\{< \cdot, \cdot >_P\}_{P \in M}$ and ∇ be the Riemannian connection, then the *curvature* R of ∇ is defined by

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z,$$

for $X, Y, Z \in C^\infty(M, \mathbb{R})$ where the *Lie bracket* is defined by $[X, Y] = XY - YX$. The *curvature tensor*

$$R(X, Y, Z, W) : M \longrightarrow \mathbb{R}$$

is defined by $R(X, Y, Z, W)(P) =< R(X, Y)Z, W >_P$ for every $P \in M$ and $X, Y, Z, W \in C^\infty(M, \mathbb{R})$.

In coordinates, if M is of coordinates $(x_1, \ldots, x_n)$, then we may write the curvature tensor as

$$R = \sum_{i,j,k,l} R_{ijkl} dx^i \otimes dx^j \otimes dx^k \otimes dx^l$$

where

$$R_{ijkl} = R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right).$$

There are properties of curvature tensor which can be checked directly.

Proposition 2.1.
(i) $R_{ijkl} = R_{klij}$;
(ii) $R_{ijkl} = -R_{ijlk}$;
(iii) $R_{ijkl} = R_{iklj} = R_{iljk}$.

Definition 2.7 (Riemannian Sectional Curvature). Let $P \in M$ and $X_P, Y_P \in T_P M$, then the *Riemannian sectional curvature* of the plane S spanned by X_P and Y_P is defined by

$$\frac{R(X, Y, X, Y)(P)}{||X_P \wedge Y_P||^2},$$

where $||X_P \wedge Y_P||^2 =< X_P, X_P >_P< X_P, X_P >_P - < X_P, Y_P >_P^2$.

The fact is that it is independent of the generators X_P, Y_P of S and this is the Gaussian curvature of the image of S under the exponential map which is a real surface through P. The Riemannian manifold M is of *positive(negative* Riemannian sectional curvature if and only if for $u \wedge v \neq 0$, $R(u, v, u, v) > 0$. Or equivalently in local coordinates (cited from [Mok] p. 31), the Riemannian manifold M is of *positive(negative)* Riemannian sectional curvature if and only if over any point and for any linear independent pair $X = \sum_i u^i \frac{\partial}{\partial x_i}$ and $Y = \sum_i v^i \frac{\partial}{\partial x_i}$,

$$R_{ijkl}(u^i v^j - v^i u^j)(u^k v^l - v^k u^l) > (<)0. \tag{2}$$

There are also definitions of Ricci curvature and scalar curvature, we will postpone them to the contexts about curvature of vector bundles.

2.3. Complex Manifolds.

DEFINITION 2.8 (Complex Manifolds). Let M be a differentiable manifold of real dimension $2n$ and if further M have an open covering $\{U_i\}_{i\in I}$ and coordinate maps

$$\phi_i : U_i \longrightarrow \mathbb{C}^n$$

such that for any pair i, j where $W = U_i \cap U_j$ is nonempty, then $\phi_i \circ {\phi_j}^{-1}$ and $\phi_j \circ {\phi_i}^{-1}$ are holomorphic on $\phi_i(W)$ and $\phi_j(W)$ respectively. Then M is called a *complex manifold of complex dimension n*

REMARK 2.2. Regarding the complex manifold as a direct generalization of the differentiable manifold by replacing $\mathbb{R}$ by $\mathbb{C}$, note the condition that $\phi_j \circ {\phi_i}^{-1}$ are *holomorphic* instead of simply differentiable. The reason is that, the unique property of complex manifolds from differentiable manifolds is that we can define *holomorphic functions* on a complex manifold: A function $f : M \longrightarrow \mathbb{C}$ is said to be *holomorphic* if for any $U_i \subset M$ the function

$$f \circ {\phi_i}^{-1} : \phi_i(U_i) \longrightarrow \mathbb{C}$$

is holomorphic in the usual sense. This definition is well-defined if and only if it gives the same answer when using different patches and coordinate functions. That is, for $x \in U_i$, $f \circ {\phi_i}^{-1}(x)$ is holomorphic if and only if $f \circ {\phi_j}^{-1}(x)$ is also holomorphic whenever U_j containing x. This is precisely the condition that $\phi_j \circ {\phi_i}^{-1}(x)$ is holomorphic, since $f \circ {\phi_i}^{-1}(x) = f \circ {\phi_j}^{-1}\phi_j \circ {\phi_i}^{-1}(x)$.

2.4. Hermitian Manifolds.

2.4.1. *Complex Structures and Complexification of Real Vector Spaces.*

DEFINITION 2.9 (Complex Structure). Let V be an n-dimensional real vector space. Let J be a linear endomorphism from V to V satisfying $J^2 = -id_V$. Such J is called a *complex structure of* V.

λ is an *eigenvalue* of the operator J if $Jx = \lambda x$ for some non-zero $x \in V$. We will look for eigenvalues of J. Suppose λ is an eigenvalue, then we have $Jx = \lambda x$ for some non-zero $x \in V$. Apply J on both sides of the equation and get

$$J^2 x = J\lambda x = \lambda J x = \lambda\lambda x = \lambda^2 x,$$

hence $\lambda^2 = -1$. However, since the image of J by definition is a real vector space, we cannot have a real vector λx with $\lambda^2 = -1$. Therefore, the real vector space V has no eigenvalues. To find eigenvalues for the J-operator, we will enlarge the space by complexifying it. Let V be a real vector space with real dimension n and $\{e_1, e_2, \ldots, e_n\}$ be a basis of V. Then the *complexification* of V, $V \otimes \mathbb{C}$ is defined by $\mathbb{C}\{e_1, e_2, \ldots, e_n\}$. Note that this complexification is independent of choices of basis of the real vector space and also note that $V \otimes \mathbb{C}$ is complex n-dimensional.

Let J be the operator on the real vector space V defined as before, we will define a correspondent $\tilde{J}$-operator on the complexification of V,

$$\tilde{J} : V \otimes \mathbb{C} \longrightarrow V \otimes \mathbb{C}$$

by $\tilde{J}(x\cdot c) = (Jx)\cdot c$ for all $x \in V$ and $c \in \mathbb{C}$. Alternatively, we will give another definition of complex structure on the complexification space and show that it is equivalent to the definition of $\tilde{J}$.

Definition 2.10 (Complex Structure of a Complex Vector Space). Let $J' : V \otimes \mathbb{C} \longrightarrow V \otimes \mathbb{C}$ a complex linear mapping such that $J'^2 = -id$. Then J' is called a *complex structure of the space* $V \otimes \mathbb{C}$.

To show that $\tilde{J}$ is a complex structure of a complex vector space, check that

$$\tilde{J}^2(v\cdot c) = J(Jv\cdot c) = J^2 v\cdot c = -v\cdot c$$

for any $v \in V$ and $c \in \mathbb{C}$. Therefore $\tilde{J}^2 = -id$ and hence $\tilde{J}$ is a complex structure of $V\otimes\mathbb{C}$. To show that J' is actually an extension of some J-operator of the real vector space V. Simply define

$$J : V \longrightarrow V$$

by $J(v) = \dfrac{J'(v\cdot c)}{c}$ for any $c \in \mathbb{C} - \{0\}$. To show that it is well defined we may show that for any other $c' \in \mathbb{C} - 0$, we have $\dfrac{J'(v\cdot c)}{c} = \dfrac{J'(v\cdot c')}{c'}$. It is true since J' is complex linear. Therefore the definition of $\tilde{J}$ and J' are equivalent.

Define $\tilde{\lambda}$ to be an *eigenvalue* of $\tilde{J}$ if there exist some nonzero vector $x\cdot c \in V\otimes\mathbb{C}$ such that $\tilde{J}(x\cdot c) = \tilde{\lambda}(x\cdot c)$. We are now looking for eigenvalues of $\tilde{J}$: Suppose $\tilde{\lambda}$ is an eigenvalue of $\tilde{J}$, then $\tilde{J}(x\cdot c) = \tilde{\lambda}(x\cdot c)$ for some non-zero vector $x\cdot c \in V\otimes\mathbb{C}$. Operate by $\tilde{J}$ on both sides of the equation and get

$$\tilde{J}\tilde{J}(x\cdot c) = \tilde{J}\tilde{\lambda}(x\cdot c) = \tilde{\lambda}\tilde{J}(x\cdot c) = \tilde{\lambda}\tilde{\lambda}(x\cdot c) = \tilde{\lambda}^2(x\cdot c).$$

On the other hand,

$$\tilde{J}\tilde{J}(x\cdot c) = J'J'(x\cdot c) = -x\cdot c$$

thus $\tilde{\lambda}^2 = -1$ and $\tilde{\lambda} = \pm\sqrt{-1}$. Since eigenvalues appear in conjugate pairs, that $\tilde{J}$ has eigenvalues on $V\otimes\mathbb{C}$ requires $V\otimes\mathbb{C}$ to be of even dimensional. Conversely, this condition is also sufficient. Under this condition i.e. n is even, we can actually get two eigenvalues of $\tilde{J}$ that is $\sqrt{-1}$ and $-\sqrt{-1}$. We can also decompose the complexified $V\otimes\mathbb{C}$ into be two eigenspaces of $\sqrt{-1}$ and $-\sqrt{-1}$ respectively, and each with complex dimension $n/2$.

2.4.2. *Decompositions of Tangent Bundles.* In the following, let M be a complex manifold with complex dimension n and P be a point in M. The local holomorphic coordinate system of M at P is $\{z_1, \ldots, z_n\}$. *Tangent bundle* TM of the n-dimensional complex manifold M is defined as the union of all tangent space T_PM of all points $P \in M$. We will introduce three notions of tangent bundles from [Griffiths].

(i) $T_P^{\mathbb{R}}M$, *Real Tangent Space of* M *at* P. Consider M as a real manifold of real dimension $2n$. Write $z_i = x_i + \sqrt{-1}y_i, i = 1, \ldots, n$, then

$$T_P^{\mathbb{R}}M = \mathbb{R}\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right\}_{i=1,\ldots,n}.$$

(ii) $T_P^{\mathbb{C}}M = T_P^{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C}$, *Complexified Tangent Space of M at P.* It is given by the complexification of the real tangent space regarded as a real vector space. That is

$$T_P^{\mathbb{C}}M = \mathbb{C}\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right\}_{i=1,\dots,n}.$$

After changing of basis by

$$\begin{cases} \frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}) \\ \frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}) \end{cases}$$

where $i = 1, \dots, n$. We may write

$$T_P^{\mathbb{C}}M = \mathbb{C}\left\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\right\}.$$

Note that $T_P^{\mathbb{C}}M$ is a complex $2n$ dimensional vector space.

(iii) $T_P^{1,0}M$ and $T_P^{0,1}M$, *Holomorphic and Antiholomorphic Tangent Spaces of M at P.* These two vector spaces are defined as the eigenspaces of the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ of the operator $\tilde{J}$ on the complexified vector space $T_P^{\mathbb{C}}M$, respectively. Therefore, each of them is of complex dimension n. We call $T_P^{1,0}M$ the *holomorphic tangent space* to M at P and $T_P^{0,1}M$ the *antiholomorphic tangent space* to M at P. Vectors in $T_P^{1,0}M$ are said to be of *type* $(1,0)$ and vectors in $T_P^{0,1}M$ are said to be of *type* $(0,1)$. We denote $T^{1,0}M = \cup_{p\in M}T_P^{1,0}M$ and $T^{0,1}M = \cup_{P\in M}T_P^{0,1}M$.

When we give a specific $J : T_P^{\mathbb{R}} \longrightarrow T_P^{\mathbb{R}}$ operator defined by

$$\begin{cases} J_P(\frac{\partial}{\partial x_j}) = & \frac{\partial}{\partial y_j} \\ J_P(\frac{\partial}{\partial y_j}) = & -\frac{\partial}{\partial x_j} \end{cases}$$

where $j = 1, \dots, n$. It can be checked by direct computation that $J_P^2 = -id_P$. Then it induces the operator on the complexified tangent space $\tilde{J} : T_P^{\mathbb{C}} \longrightarrow T_P^{\mathbb{C}}$. It can be checked that $\dfrac{\partial}{\partial z_i}\cdot c$ and $\dfrac{\partial}{\partial \bar{z}_i}\cdot c$ for any $c \in \mathbb{C}-\{0\}$are the corresponding eigenvectors. That is $\tilde{J}(\dfrac{\partial}{\partial z_i}\cdot c) = \sqrt{-1}\dfrac{\partial}{\partial z_i}\cdot c$ and $\tilde{J}(\dfrac{\partial}{\partial \bar{z}_i}\cdot c) = -\sqrt{-1}\dfrac{\partial}{\partial \bar{z}_i}\cdot c$. From now on, we will denote $\tilde{J}$ by J if there is no ambiguity.

REMARK 2.3. Observe that

$$\overline{\left(\frac{\partial}{\partial z_i}\right)} = \overline{\frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right)} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial \bar{z}_i}.$$

Hence

$$T_P^{0,1}M = \mathbb{C}\left\{\frac{\partial}{\partial \bar{z}_i}\right\} = \mathbb{C}\left\{\overline{\left(\frac{\partial}{\partial z_i}\right)}\right\} = \overline{\mathbb{C}\left\{\frac{\partial}{\partial z_i}\right\}} = \overline{T_P^{1,0}M}.$$

Therefore, there is an $\mathbb{R}$-isomorphism $\pi \circ f$ from $T_P^{\mathbb{R}}M$ to $T_P^{1,0}M$ defined by

$$T_P^{\mathbb{R}}M \xrightarrow{f} T_P^{\mathbb{C}}(M) = T_P^{1,0}M \oplus \overline{T_P^{1,0}M} \xrightarrow{\pi} T_P^{1,0}.$$

where π is the projection map and

$$\begin{aligned} f(v) &= a\frac{\partial}{\partial x_i} + b\frac{\partial}{\partial y_i} \\ &= a\left(\frac{\partial}{\partial z_i} + \frac{\partial}{\partial \bar{z}_i}\right) + b(-\sqrt{-1})\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial \bar{z}_i}\right) \\ &= (a - \sqrt{-1}b)\frac{\partial}{\partial z_i} + (a + \sqrt{-1}b)\frac{\partial}{\partial \bar{z}_i} \end{aligned}$$

for any $v = a\frac{\partial}{\partial x_i} + b\frac{\partial}{\partial y_i} \in T_P^{\mathbb{R}}M$. Since $\pi \circ f(v) = (a - \sqrt{-1}b)\frac{\partial}{\partial z_i}$, then $\pi \circ f$ is an $\mathbb{R}$-isomorphism. In this way, the real tangent space is sometimes identified with the holomorphic tangent space.

2.4.3. *Hermitian Metrics and Hermitian Manifolds.*

Definition 2.11 (Hermitian Manifolds). Let M be a complex manifold with complex dimension n, then it can be regarded as a real $2n$ dimensional differentiable manifold. Let g be a Riemannian metric defined on this real differentiable manifold. Hence (M, g) is a Riemannian manifold. Let J be a complex structure on the tangent space at each point. g is called J-invariant if $g(u, v) = g(Ju, Jv)$ for all $u, v \in T_P^{\mathbb{R}}M$, M associated with a J-invariant metric g is called a *Hermitian manifold*. This J-invariant metric g is then called a *Hermitian metric* of M

We may extend $g : T_P^{\mathbb{R}}M \times T_P^{\mathbb{R}}M \longrightarrow \mathbb{R}$ by complex bi-linearity to

$$\tilde{g} : (T_P^{\mathbb{R}}M \otimes \mathbb{C}) \times (T_P^{\mathbb{R}}M \otimes \mathbb{C}) \longrightarrow \mathbb{C}.$$

Precisely, $\tilde{g}(u \cdot c, v \cdot d) = c \cdot d \cdot g(u, v)$ for $u, v \in T_P^{\mathbb{R}}M$ and $c, d \in \mathbb{C}$. Also extend J to

$$\tilde{J} : T^{\mathbb{R}}M \otimes \mathbb{C} \longrightarrow T^{\mathbb{R}}M \otimes \mathbb{C}$$

as we did in Subsection (2.4.1). Therefore, g is J-invariant if and only if $\tilde{g}$ is $\tilde{J}$-invariant. In the following context, we will denote $\tilde{g}$ by g and $\tilde{J}$ by J if there arises no ambiguity. Check by direct computation, we have:

Proposition 2.4. *The condition of g to be J-invariant is equivalent to the condition that $g(u \cdot c, v \cdot d) = 0$ for $u \cdot c$ and $v \cdot d$ of the same type, i.e. the eigenvectors of the same eigenvalues.*

Decompose a metric g on $T^{\mathbb{C}}M \times T^{\mathbb{C}}M$ into four parts as follow:

$$\begin{cases} \text{(a) } g : T_P^{1,0}M \times T_P^{1,0}M \longrightarrow \mathbb{C} \\ \text{(b) } g : T_P^{1,0}M \times T_P^{0,1}M \longrightarrow \mathbb{C} \\ \text{(c) } g : T_P^{0,1}M \times T_P^{1,0}M \longrightarrow \mathbb{C} \\ \text{(d) } g : T_P^{0,1}M \times T_P^{0,1}M \longrightarrow \mathbb{C} \end{cases}$$

(a) and (d) are zero mappings. Since g is complex extension of a Riemannian metric, then we have $g(u, v) = g(v, u)$, i.e. (b) and (c) gives the same metric. Therefore, (b) gives all the information of the metric g. Recall that if we choose the J-operator properly, that is we define it by $J\left(\frac{\partial}{\partial x}\right) = -\frac{\partial}{\partial y}$ and $J\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x}$ we will have $T_P^{1,0}M = \overline{T_P^{0,1}M}$ which means $\overline{\left(\frac{\partial}{\partial z_i}\right)} = \frac{\partial}{\partial \bar{z}_i}$ for all $i = 1, 2, \ldots, n$. From now on we will adopt this particular J-operator. Hence g acts the same on $T^{1,0} \times T^{0,1}$ as on $\overline{T^{0,1}} \times \overline{T^{1,0}}$. Thus we have $g(\eta, \xi) = \overline{g(\xi, \eta)}$ for case (b). From the above observation and under the coordinates we have chosen, the Hermitian metric can be written in the form

$$\begin{aligned} g &= \sum g_{i\bar{j}} dz^i \otimes d\bar{z}^j + g_{\bar{i}j} d\bar{z}^i \otimes dz^j \\ &= 2Re \sum g_{i\bar{j}} dz^i \otimes d\bar{z}^j \end{aligned}$$

Only consider the quarter $T_P^{1,0}M \times \overline{T_P^{1,0}M}$ of the whole tangent space, we have the restriction of Hermitian metrics on it. Let $\{\frac{\partial}{\partial z_i}\}_{i=1,\ldots,n}$ be holomorphic basis of $T_P^{1,0}M$, define a metric

$$(\quad,\quad)_P : T_P^{1,0}M \times \overline{T_P^{1,0}M} \longrightarrow \mathbb{C}$$

by $h_{ij} = (\frac{\partial}{\partial z_i}, \overline{\frac{\partial}{\partial z_j}}) = g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})$. Let $\{dz^i\}_{i=1,\ldots,n}$ be the dual basis of $\{\frac{\partial}{\partial z_i}\}_{i=1,\ldots,n}$ and $\{d\bar{z}^i\}_{i=1,\ldots,n}$ be the dual basis of $\{\frac{\partial}{\partial \bar{z}_i}\}_{i=1,\ldots,n}$. Then the *Hermitian form* is defined by

$$H = \sum_{i,j=1}^{n} h_{ij} dz^i \otimes d\bar{z}^j.$$

2.5. Kähler Manifolds. Let (M, g) be a Hermitian manifold,

$$\omega = \sqrt{-1} \sum g_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

is called the *fundamental (1, 1)-form* of M. Simple calculation gives $\omega = -2$ Imaginary part of $\left(\sum g_{i\bar{j}} dz^i \otimes d\bar{z}^j\right)$, hence ω is a real form.

Definition 2.12 (Kähler Manifolds and Kähler Metrics). A Hermitian manifold (M, g) is called a *Kähler manifold* if its fundamental $(1,1)$-form is d-closed, and the corresponding Hermitian metric is then called a *Kähler metric.*

2.6. Example: Complex Projective Space $\mathbb{P}^n_{\mathbb{C}}$. Consider the $(n+1)$ complex dimensional vector space $\mathbb{C}^{n+1} - \{0\}$, where 0 denotes the origin. In $\mathbb{C}^{n+1} - \{0\}$, identify the points on the same line through the origin. This gives an equivalence relation $\sim$. In local coordinates under standard basis $\{e_0, \ldots, e_n\}$, let $z = (z_0, \ldots, z_n)$ and $w = (w_0, \ldots, w_n)$ be two points in $\mathbb{C}^{n+1} - \{0\}$, then $(z_0, \ldots, z_n) \sim (w_0, \ldots, w_n)$ if and only if there exists some non-zero constant $\lambda \in \mathbb{C}$ s.t. $(z_0, \ldots, z_n) = \lambda(w_0, \ldots, w_n)$. The *complex projective space* $\mathbb{P}^n_{\mathbb{C}}$ is defined to be the quotient space $\mathbb{C}^{n+1} - \{0\}/\sim$. (i) Show that $\mathbb{P}^n_{\mathbb{C}}$ is a complex manifold. Let $U_i = \{(z_0, \ldots, z_i, \ldots, z_n) \in \mathbb{C}^{n+1} : z_i \neq 0\}$ where $i = 0, \ldots, n$.

Observe that each U_i is homeomorphic to $\mathbb{C}^n$ and this $(n+1)$ open sets $\{U_i\}$ cover $\mathbb{P}^n_{\mathbb{C}}$. In each U_i, we have the local coordinates ${}_i\zeta^k = \frac{z_k}{z_i}, 0 \le k \le n, k \ne i$. The transition of local coordinates in $U_i \cap U_j$ is given by ${}_j\zeta^h = \frac{{}_i\zeta^h}{{}_i\zeta^j}, 0 \le h \le n, h \ne j$, which are holomorphic functions. It remains to show that the Jacobian matrix is non-singular. We will calculate the determinant of the Jacobian matrix. For fixed i, j, write the matrix as $A = (A_{kh})_{0 \le h,k \le m, h \ne j, k \ne i}$ with $A_{kh} = \frac{\partial_j \zeta^h}{\partial_i \zeta^k}$. Compute $A_{kh} = \frac{\partial_j \zeta^h}{\partial_i \zeta^k} = \frac{\delta^h_k \cdot_i \zeta^j -_i \zeta^h \delta^j_k}{({}_i\zeta^j)^2}$. Thus $detA = \left(\frac{1}{{}_i\zeta^j}\right)^n \ne 0$. Therefore, $\mathbb{P}^n_{\mathbb{C}}$ is a complex manifold.

(ii) Define the *Fubini-Study metric* on $\mathbb{P}^n_{\mathbb{C}}$. We will define g on the set U_0 first, and then show that it is in general well defined on M. Define $g_{FS} = \sum g_{i\bar{j}} dz^i \otimes d\bar{z}^j$ by

(3) $$g_{i\bar{j}} = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log(1 + |z_1|^2 + \cdots + |z_n|^2).$$

To prove the metric is globally defined, write $U_1 = \{[w_0, 1, w_2, \ldots, w_n]\}$, then $z_i = \frac{w_i}{w_0}$ for all $i \ne 1$ and $z_1 = \frac{1}{w_0}$ and so

$$\begin{aligned} g_{FS} &= \sum \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log(1 + |z_1|^2 + \cdots + |z_n|^2) dz^i \otimes d\bar{z}^j \\ &= \sum \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log\left(1 + \frac{1}{|w_0|^2} + \frac{|w_2|^2}{|w_0|^2} + \cdots + \frac{|w_n|^2}{|w_0|^2}\right) dw^i \otimes d\bar{w}^j \\ &= \sum \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \left(\log(1 + |w_2|^2 + \cdots + |w_n|^2) - \log|w_0|^2\right) dw^i \otimes d\bar{w}^j \\ &= \sum \frac{\partial^2}{\partial w_i \partial \bar{w}_j} \log(1 + |w_2|^2 + \cdots + |w_n|^2) dw^i \otimes d\bar{w}^j \end{aligned}$$

The last equality is because w_0 is holomorphic. So g_{FS} is globally defined and the corresponding metric on $\mathbb{P}^n_{\mathbb{C}}$ is called the *Fubini-Study metric*. The reference of this example is from [Tian]. It can be shown to be a Kähler metric.

3. Sheaves and Čech Cohomology

We will first give definitions of presheaf, sheaf and their relation and then give the definition of Čech cohomology theory. The reference used is [Hir].

3.1. Definition of Sheaves and Presheaves.

Definition 3.1 (Sheaf). A *sheaf* $\mathfrak{S}$ (of abelian groups) over X is a triple $\mathfrak{S} = (S, \pi, X)$ which satisfies the following three properties:

(i) S and X are topological spaces, π is a continuous and surjective map from S to X.

(ii) Every point $\alpha \in S$ has an open neighborhood U in S which is homeomorphic to an open neighborhood V of $\pi(\alpha)$ in X.
(iii) Let $x \in X$, we call the inverse image of x in S, $\pi^{-1}(x) \in S$ the *stalk* of x, then $S_x, x \in X$ is an abelian group with an additive operation $-$. Further, $\alpha - \beta$ is continuous with respect to α and β.

Definition 3.2 (Presheaf). A *presheaf* $\mathfrak{G} = \{S_U, r_V^U\}$ over X consists of an abelian group S_U for each open set U of X and a homomorphism $r_V^U : S_U \longrightarrow S_V$ for each pair of open sets U, V of X with $V \subset U$. These groups and homomorphisms satisfies the following properties:

(i) If U is empty then $S_U = 0$ is the trivial group.
(ii) The homomorphism $r_U^U : S_U \longrightarrow S_U$ is the identity. If $W \subset V \subset U$, then $r_W^U = r_W^V r_V^U$.

Sheaf and presheaf are related. A sheaf can be gotten from a presheaf $\mathfrak{G} = \{S_U, r_V^U\}$ by means of taking direct limit. Let $x \in X$ and U and V be two open sets both containing x in X, we say an element $g_U \in S_U$ and $g_V \in S_V$ are equivalent if and only if there exists an open neighborhood $W \subset U \cap V$ containing x such that $r_W^U g_U = r_W^V g_V$ on W. It can be checked that the relation $\sim$ does give reflexitivity, symmetry, and transitivity. Denote the equivalence class at x by g_x. i.e. $g_x = [g_U] = \{g_V : g_U \sim g_V, U, V \text{are open sets containing } x\}$ and call the equivalence class a *germ at point* x or *direct limit of* g_U, U *containing* x, *at the point* x. The process of getting the equivalence classes called *taking direct limit*. Define $S_x = \{g_x\}$ be the set of all germs at x, $S = \{S_x, x \in X\}$ and a mapping $\pi : S \longrightarrow X$ which maps S_x to $x \in X$. Then the triple $\mathfrak{S} = (S, \pi, x)$ will define a sheaf. The proof is by checking the axioms of sheaf for $\mathfrak{S}$ directly. $\mathfrak{S}$ is called the *corresponding sheaf of the presheaf* $\mathfrak{G}$.

Conversely, a presheaf can be constructed from a sheaf. Let $\mathfrak{S} = (S, \pi, X)$ be a sheaf, let U be an open set in X, a *section* is defined by a continuous map $s : U \longrightarrow S$ such that $\pi \circ s = id_U$. Denote $\Gamma(U, \mathfrak{S})$ the set of all sections of the open set U. $\Gamma(U, \mathfrak{S})$ is an abelian group with the additive operation $-$ induced from that of $S_x, x \in X$ in a natural way and the zero element of the abelian group is the zero section $s_0 : x \longrightarrow 0_x, x \in X$. In particular, if U is empty, $\Gamma(U, \mathfrak{S})$ is defined to be the trivial group. Associate each open set U of X an abelian group $\Gamma(U, \mathfrak{S})$. Then $\mathfrak{G} = \{\Gamma(U, \mathfrak{S}), r_V^U\}$ where r_V^U is the restriction map, will be a presheaf. The proof is by checking the axioms for presheaf for $\mathfrak{G}$ directly. $\mathfrak{G}$ is called the *canonical presheaf of the sheaf* $\mathfrak{S}$.

3.2. Homomorphisms between (Pre)Sheaves.

3.2.1. *Homomorphisms between Presheaves.* Let $\mathfrak{G} = \{S_U, r_V^U\}$ and $\tilde{\mathfrak{G}} = \{\tilde{S}_U, \tilde{r}_V^U\}$ be two presheaves over the topological space X. We define a homomorphism

$$h : \mathfrak{G} \longrightarrow \tilde{\mathfrak{G}}$$

by the set of all ordinary group homomorphism from S_U to $\tilde{S}_U$ for any open set W of X. To get rid of the ambiguity which may occur when taking the restriction map, the following diagram should also commute:

$$\begin{array}{ccc} S_U & \xrightarrow{\text{h}} & \tilde{S}_U \\ {\scriptstyle \mathrm{r}_{\mathrm{V}}^{\mathrm{U}}}\downarrow & & \downarrow{\scriptstyle \tilde{\mathrm{r}}_{\mathrm{V}}^{\mathrm{U}}} \\ S_V & \xrightarrow{\text{h}} & \tilde{S}_V \end{array}$$

3.2.2. *Homomorphisms between Sheaves.* Homomorphisms between sheaves can be induced by homomorphisms between presheaves. Let $\mathfrak{S} = (S, \pi, X)$ and $\tilde{\mathfrak{S}} = (\tilde{S}, \pi, X)$ be two sheaves with canonical presheaves $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$ respectively. To define a homomorphism between the two sheaves it suffices to define a map between S_x to $\tilde{S}_x$, $x \in X$. If h is a homomorphism from $\mathfrak{G}$ to $\tilde{\mathfrak{G}}$ then we define the homomorphism h_* from $\mathfrak{S}$ to $\tilde{\mathfrak{S}}$ by taking direct limit of the following commutative diagram:

$$\begin{array}{ccc} & \vdots & \\ S_U & \xrightarrow{\text{h}} & \tilde{S}_U \\ {\scriptstyle r_V^U}\downarrow & & \downarrow{\scriptstyle \tilde{r}_V^U} \\ S_V & \xrightarrow{\text{h}} & \tilde{S}_V \\ {\scriptstyle r_V^U}\downarrow & & \downarrow{\scriptstyle \tilde{r}_V^U} \\ S_W & \xrightarrow{\text{h}} & \tilde{S}_W \\ & \vdots & \\ S_x & \xrightarrow{\text{h}*} & \tilde{S}_x. \end{array}$$

That is we define $h_* : S_x \longrightarrow \tilde{S}_x$ by the direct limit of $h(S_U)$.

To show that this definition is independent of the choices of representatives in S_x. Let S_V be another representative of S_x, then there exists an open set $W \subset U \cap V$ containing x such that $r_W^U h = r_W^V h$ on W. Hence hS_V and hS_U are equivalent. Therefore the homomorphism h_* is well-defined.

3.3. From Sheaf to Presheaf and from Presheaf to Sheaf. To complete the picture of the relation between sheaf and presheaf, we will show that the corresponding sheaf of a canonical presheaf of a sheaf is the sheaf itself. Let $\mathfrak{G} = \{S_U, r_V^U\}$ be a presheaf over X and $\mathfrak{S} = (S, \pi, X)$ be the corresponding sheaf. Let $\tilde{\mathfrak{G}} = \{\tilde{S}_U, \tilde{r}_V^U\}$ be the canonical presheaf of the sheaf $\mathfrak{S}$ and $\tilde{\mathfrak{S}}$ be the corresponding sheaf of the presheaf $\tilde{\mathfrak{G}}$.

THEOREM 3.1. $\mathfrak{S} = \tilde{\mathfrak{S}}$.

Proof. Consider the following commutative diagram,

$$
\begin{array}{ccc}
 & \vdots & \\
S_U & \xrightarrow{\text{h}} & \Gamma(U, \mathfrak{S}) \\
r_V^U \big\downarrow & & \big\downarrow \tilde{r}_V^U \\
S_V & \xrightarrow{\text{h}} & \Gamma(V, \mathfrak{S}) \\
r_V^U \big\downarrow & & \big\downarrow \tilde{r}_V^U \\
S_W & \xrightarrow{\text{h}} & \Gamma(W, \mathfrak{S}) \\
 & \vdots & \\
\mathfrak{S} & \xrightarrow{\text{h}*} & \tilde{\mathfrak{S}}.
\end{array}
$$

We define the homomorphisms h from S_U to $\Gamma(U, \mathfrak{S})$ for any U containing $x \in X$ by $h(g_U) = s$ where s is the section of U such that s maps each point in U to the equivalence class with representative determined by the value of g_U at that point. Therefore the induced homomorphism h_* of sheaves is the identity map. □

3.4. Cohomology Groups $H^q(\mathfrak{A}, \boldsymbol{\mathfrak{G}})$ and $H^q(\mathfrak{A}, \mathfrak{S})$. Let $\mathfrak{A} = \{U_i\}_{i \in I}$ be an open covering of X and $\boldsymbol{\mathfrak{G}} = \{S_U, r_V^U\}$ be a presheaf over X. A *q-cochain* is a function f which associates to each q-term indices $i_1, i_2, \ldots, i_q \in I$ an element $f(i_1, i_2, \ldots, i_q)$ of $S_{U_{i_1, i_2, \ldots, i_q}}$ where $U_{i_1, i_2, \ldots, i_q}$ denotes $U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_q}$. The group of all q-cochains is denoted as $C^q(\mathfrak{A}, \boldsymbol{\mathfrak{G}})$.

Definition 3.3 (Coboundary Homomorphism). The coboundary homomorphism

$$\delta^q : C^q(\mathfrak{A}, \boldsymbol{\mathfrak{G}}) \longrightarrow C^{q+1}(\mathfrak{A}, \boldsymbol{\mathfrak{G}})$$

is defined by $\delta^q(f)(i_1, i_2, \ldots, i_{q+1}) = \sum_{k=1}^{q+1} r_W^{W_k} (-1)^k f(i_1, \ldots, \hat{i_k}, i_{q+1}) \in C^{q+1}(\mathfrak{A}, \boldsymbol{\mathfrak{G}})$ where $W = U_{i_1, i_2, \ldots, i_{q+1}}$ and $W_k = U_{i_1, \ldots, \hat{i_k}, \ldots, i_{q+1}}$ for $f \in C^q(\mathfrak{A}, \boldsymbol{\mathfrak{G}})$.

Proposition 3.1. *$\delta^{q+1}\delta^q = 0$ for all $q > 0$.*

Proof. Direct computation. □

Let f be a q-cochain in $C^q(\mathfrak{A}, \boldsymbol{\mathfrak{G}})$. If $\delta^q(f) = 0$, then f is called a *q-cocycle*. If there exists a $(q-1)$-cochain g such that $f = \delta^{q-1}(g)$, then f is called a *q-coboundary*. Denote

$$Z^q(\mathfrak{A}, \boldsymbol{\mathfrak{G}}) = \{q\text{-cocycls in } C^q(\mathfrak{A}, \boldsymbol{\mathfrak{G}})\};$$

$$B^q(\mathfrak{A}, \boldsymbol{\mathfrak{G}}) = \{q\text{-coboundaries in } C^q(\mathfrak{A}, \boldsymbol{\mathfrak{G}})\}.$$

By Proposition 3.1, the following module is well-defined.

Definition 3.4 (Čech Cohomology Group). Define

$$H^q(\mathfrak{A}, \boldsymbol{\mathfrak{G}}) = Z^q(\mathfrak{A}, \boldsymbol{\mathfrak{G}}) / B^q(\mathfrak{A}, \boldsymbol{\mathfrak{G}})$$

to be the *q-th cohomology group of $\mathfrak{A}$ with coefficients in $\boldsymbol{\mathfrak{G}}$*. Define the *$q$-th cohomology of $\mathfrak{A}$ with coefficient in the sheaf $\mathfrak{S}$*, $H^q(\mathfrak{A}, \mathfrak{S})$ to be the cohomology group of its canonical presheaf $\boldsymbol{\mathfrak{G}}$, $H^q(\mathfrak{A}, \boldsymbol{\mathfrak{G}})$.

3.5. Cohomology Groups $H^q(X, \mathfrak{G})$ and $H^q(X, \mathfrak{S})$. $H^q(X, \mathfrak{G})$ is defined by taking direct limit of $H^q(\mathfrak{A}, \mathfrak{G})$:

Step 1. Define homomorphisms between cohomology groups over an open covering and its refinement. Let $\mathfrak{V} = \{V_j\}_{j\in J}$ be a refinement of the open covering $\mathfrak{A} = \{U_i\}_{i\in I}$. Choose a mapping $\tau : J \longrightarrow I$ such that $U_{\tau(j)} \supset V_j$. Thus we can induce a mapping

$$\tau^* : C^q(\mathfrak{A}, \mathfrak{G}) \longrightarrow C^q(\mathfrak{V}, \mathfrak{G})$$

by defining

$$\tau^*(f)(j_0, j_1, \ldots, j_q) = r_W^{W'} f(\tau j_0, \tau j_1, \ldots, \tau j_q)$$

where $W' = U_{\tau j_o, \tau j_1, \ldots, \tau j_q}$ and $W = V_{j_0, j_1, \ldots, j_q}$. If the homomorphism thus defined make the following diagram commute, then we can induce homomorphisms between cohomology groups by it.

Proposition 3.2. *The following diagram commute for any $q \in \mathbb{Z}^+$:*

$$\begin{array}{ccc}
 & \vdots & \\
C^{q-1}(\mathfrak{A}, \mathfrak{G}) & \xrightarrow{\tau^*} & C^{q-1}(\mathfrak{V}, \mathfrak{G}) \\
\downarrow{\scriptstyle \delta^{q-1}} & & \downarrow{\scriptstyle \delta^{q-1}} \\
C^{q}(\mathfrak{A}, \mathfrak{G}) & \xrightarrow{\tau^*} & C^{q}(\mathfrak{V}, \mathfrak{G}) \\
\downarrow{\scriptstyle \delta^{q}} & & \downarrow{\scriptstyle \delta^{q}} \\
C^{q+1}(\mathfrak{A}, \mathfrak{G}) & \xrightarrow{\tau^*} & C^{q+1}(\mathfrak{V}, \mathfrak{G}) \\
 & \vdots &
\end{array}$$

Proof. Direct checking. □

Define homomorphism

$$\tau^* : H^q(\mathfrak{A}, \mathfrak{G}) \longrightarrow H^q(\mathfrak{V}, \mathfrak{G})$$

by letting $\tau^*(f + B^q(\mathfrak{A}, \mathfrak{G})) = \tau^*(f) + \tau^*(B^q(\mathfrak{A}, \mathfrak{G}))$ where $f \in Z^q(\mathfrak{A}, \mathfrak{G})$ is a representative of an element in the cohomology group $H^q(\mathfrak{A}, \mathfrak{G})$ and τ^* on the right hand side of the identity is the homomorphism between cochains. It remains to show that $\tau^*(B^q(\mathfrak{A}, \mathfrak{G})) = B^q(\mathfrak{V}, \mathfrak{G})$: for any $g \in B^q(\mathfrak{A}, \mathfrak{G})$, let $g = \delta^{q-1}g'$ for some $g' \in C^{q-1}(\mathfrak{A}, \mathfrak{G})$, then

$$\tau^* g = \tau^* \delta^{q-1} g' = \delta^{q-1} \tau^* g'.$$

Note that $\tau^* g' \in C^{q-1}(\mathfrak{V}, \mathfrak{G})$ we have shown that $\tau^* g \in B^q(\mathfrak{V}, \mathfrak{G})$. Therefore, $\tau^*(B^q(\mathfrak{A}, \mathfrak{G})) \subset B^q(\mathfrak{V}, \mathfrak{G})$. Conversely, for any $f \in B^q(\mathfrak{V}, \mathfrak{G})$, let $f = \delta^{q-1} f'$ for some $f' \in C^{q-1}(\mathfrak{V}, \mathfrak{G})$. Let $g' \in C^{q-1}(\mathfrak{A}, \mathfrak{G})$ satisfying $\tau^* g' = f'$ we can always find such g' since $\mathfrak{V}$ is a refinement of $\mathfrak{A}$. Thus

$$\delta^{q-1} \tau^* g' = \tau^* \delta^{q-1} g' = \tau^* g.$$

Note that $g \in C^q(\mathfrak{A}, \mathfrak{G})$ thus $B^q(\mathfrak{V}, \mathfrak{G}) \subset \tau^* B^q(\mathfrak{A}, \mathfrak{G})$. Therefore, by using the commutativity of the diagram of cochains, we have defined the induced homomorphism between cohomology groups. It can be checked that the homomorphism is actually independent of the mapping $\tau : J \longrightarrow I$ chosen at the beginning. Hence it is well-defined. We denote it simply by $t^{\mathfrak{A}}_{\mathfrak{V}}$ by only referring to the covering.

Step 2. Realize the process of taking direct limit as follows:

$$H^q(\mathfrak{A}, \mathfrak{G}) \xrightarrow{t^{\mathfrak{A}}_{\mathfrak{V}}} H^q(\mathfrak{V}, \mathfrak{G}) \xrightarrow{t^{\mathfrak{V}}_{\mathfrak{W}}} H^q(\mathfrak{W}, \mathfrak{G}) \longrightarrow \cdots H^q(X, \mathfrak{G})$$

We need to consider the chain of refinements of the open coverings.

LEMMA 3.3. *$t^{\mathfrak{A}}_{\mathfrak{A}}$ is identity and $t^{\mathfrak{A}}_{\mathfrak{W}} = t^{\mathfrak{V}}_{\mathfrak{W}} t^{\mathfrak{A}}_{\mathfrak{V}}$ whenever $\mathfrak{A}$ is an open covering of X, $\mathfrak{V}$ is a refinement of $\mathfrak{A}$ and $\mathfrak{W}$ is a refinement of $\mathfrak{V}$.*

PROOF. The first identity is trivially true. Suppose I, J and K are the index set of the open coverings $\mathfrak{A}, \mathfrak{V}$ and $\mathfrak{W}$ respectively. We may choose $\nu : K \longrightarrow J$ and $\tau : J \longrightarrow I$, define $\gamma : K \longrightarrow I$ by $\gamma = \tau \circ \nu$. We can define $t^{\mathfrak{A}}_{\mathfrak{W}}$ by γ, therefore, for any $[f] \in H^q(\mathfrak{A}, \mathfrak{G})$, we have $t^{\mathfrak{A}}_{\mathfrak{W}}[f] = [\gamma^*(f)] = [(\tau\nu)^*(f)]$ on the other hand $t^{\mathfrak{V}}_{\mathfrak{W}} t^{\mathfrak{A}}_{\mathfrak{V}}[f] = t^{\mathfrak{V}}_{\mathfrak{W}}[\tau^* f] = [\nu^* \tau^* f]$ is also get from the covering $\mathfrak{A}$ to $\mathfrak{W}$ therefore we get the composition. □

If we consider the presheaf $\{H^q(\mathfrak{A}, \mathfrak{G}), r^{\mathfrak{A}}_{\mathfrak{V}}\}$ formed by assigning each open covering of X a group $H^q(\mathfrak{A}, \mathfrak{G})$, taking direct limit of these sequence we get an equivalence class in which two elements are equivalent if there is a refinement of both of the coverings such that the q-th cohomology classes of that covering are the same. We define this equivalence class to be $H^q(X, \mathfrak{G})$ and this is the *q-th cohomology group of X with coefficients in* $\mathfrak{G}$. Let $\mathfrak{S}$ be a sheaf over X with $\mathfrak{G}$ as the canonical presheaf of it. Then we define the *q-th cohomology group of X with coefficients in the sheaf* $\mathfrak{S}$, $H^q(X, \mathfrak{S})$ to be $H^q(X, \mathfrak{G})$.

3.6. Exact Cohomology Sequence for Presheaf. Step 1. We will induce homomorphisms between cochains with coefficients in different presheaves. Recall that we have induced the homomorphisms between different presheaves by taking direct limit of the homomorphisms between different groups determined by the presheaves respectively. Consider a homomorphism $h : \mathfrak{G} \longrightarrow \mathfrak{G}'$ between presheaves over X. We would like to induce a homomorphism for any $q \in \mathbb{Z}^+$ and any covering $\mathfrak{A}$ of X:

$$h_* : C^q(\mathfrak{A}, \mathfrak{G}) \longrightarrow C^q(\mathfrak{A}, \mathfrak{G}')$$

defined by $(h_* f)(i_0, i_1, \ldots, i_q) = h(f(i_0, i_1, \ldots, i_q))$ for any $f \in C^q(\mathfrak{A}, \mathfrak{G})$.

Step 2. From an exact sequence,

$$0 \longrightarrow \mathfrak{G}' \xrightarrow{h'} \mathfrak{G} \xrightarrow{h} \mathfrak{G}'' \longrightarrow 0$$

of presheaves over X, we can induce an exact sequence for each $q > 0$

$$0 \longrightarrow C^q(\mathfrak{A}, \mathfrak{G}') \xrightarrow{h'_*} C^q(\mathfrak{A}, \mathfrak{G}) \xrightarrow{h_*} C^q(\mathfrak{A}, \mathfrak{G}'') \longrightarrow 0.$$

Step 3. By the induced homomorphism between cochains with coefficients in different presheaves, we will induce homomorphism between cohomology groups with coefficients in different presheaves. To achieve this, we have to show that the induced homomorphism commutes with the coboundary operator first.

LEMMA 3.4. *Consider the diagram for any* $q \geq 0$.

$$\begin{array}{ccc} C^q(\mathfrak{A}, \mathfrak{G}) & \xrightarrow{\mathrm{h}_*} & C^q(\mathfrak{A}, \mathfrak{G}') \\ \delta^{\mathrm{q}} \downarrow & & \downarrow \delta^{\mathrm{q}} \\ C^{q+1}(\mathfrak{A}, \mathfrak{G}) & \xrightarrow{\mathrm{h}} & C^{q+1}(\mathfrak{A}, \mathfrak{G}'), \end{array}$$

we have $\delta^q h_* = h_* \delta^q$.

PROOF. Direct checking. □

By Lemma 3.4, we can induce

$$h_* : H^q(\mathfrak{A}, \mathfrak{G}) \longrightarrow H^q(\mathfrak{A}, \mathfrak{G}')$$

by $h_*[f] = [h_* f] \in H^q(\mathfrak{A}, \mathfrak{G}')$ for any $[f] = f + B^q(\mathfrak{A}, \mathfrak{G}), f \in Z^q(\mathfrak{A}, \mathfrak{G})$ where $h_* f$ as a representative of $[h_* f]$ is in $Z^q(\mathfrak{A}, \mathfrak{G}')$.

Step 4. We will induce a long exact sequence

$$\begin{array}{rl} & 0 \longrightarrow H^0(\mathfrak{A}, \mathfrak{G}') \xrightarrow{h'_*} H^0(\mathfrak{A}, \mathfrak{G}) \xrightarrow{h_*} H^0(\mathfrak{A}, \mathfrak{G}'') \xrightarrow{\delta^0_*} H^1(\mathfrak{A}, \mathfrak{G}') \\ \cdots \longrightarrow & H^q(\mathfrak{A}, \mathfrak{G}') \xrightarrow{h'_*} H^q(\mathfrak{A}, \mathfrak{G}) \xrightarrow{h_*} H^q(\mathfrak{A}, \mathfrak{G}'') \xrightarrow{\delta^q_*} H^{q+1}(\mathfrak{A}, \mathfrak{G}') \longrightarrow \cdots \end{array}$$

where the *connection operators*

$$\delta^q_* : H^q(\mathfrak{A}, \mathfrak{G}'') \longrightarrow H^{q+1}(\mathfrak{A}, \mathfrak{G}')$$

are defined as follows: For any $[f''] \in H^q(\mathfrak{A}, \mathfrak{G}'')$ where $f'' \in Z^q(\mathfrak{A}, \mathfrak{G}'')$, by the exactness of the sequence of cochains in step 2, h_* is surjective, that is there exists an $f \in C^q(\mathfrak{A}, \mathfrak{G})$ such that $f'' = h_* f$. By the commutativity of the homomorphism and the coboundary operator again, we have $f'' \in Z^q(\mathfrak{A}, \mathfrak{G})$. By $Kerh_* = Imh'_*$, we can find an $f' \in C^q(\mathfrak{A}, \mathfrak{G}')$ such that $h'_* f' = f$. Again by the commutativity of the homomorphism and the coboundary operator, we have this $f' \in Z^q(\mathfrak{A}, \mathfrak{G}')$.

Simply define

$$\delta^q_*[f''] = [\delta^q(f')].$$

By commutativity again, we have $h'_* \delta^q(f') = \delta^q h'_*(f') = \delta^q(0) = 0$ i.e. $\delta^q(f') \in H^{q+1}(\mathfrak{A}, \mathfrak{G}')$. The exactness of the long exact sequence can be checked directly.

Step 5. Consider the sequence of cohomology groups with coefficients in same cohomology group but with different open coverings of X. Consider a sequence of open coverings, $\ldots, \mathfrak{A}, \mathfrak{V}, \mathfrak{W}, \ldots$ such that the later one is a refinement of the previous one. We can define the cohomology groups of X with coefficients in the particular presheaf, $H^q(X, \mathfrak{G})$. To induce the homomorphism

$$h_* : H^q(X, \mathfrak{G}) \longrightarrow H^q(X, \mathfrak{G}),$$

the following diagram should commute:

$$\begin{array}{ccccccccc}
H^q(\mathfrak{A},\mathfrak{G}) & \xrightarrow{t^{\mathfrak{A}}_{\mathfrak{V}}} & H^q(\mathfrak{V},\mathfrak{G}) & \xrightarrow{t^{\mathfrak{V}}_{\mathfrak{W}}} & H^q(\mathfrak{W},\mathfrak{G}) & \longrightarrow & \cdots & \longrightarrow & H^q(X,\mathfrak{G}) \\
{\scriptstyle h_*}\big\downarrow & & \big\downarrow{\scriptstyle h_*} & & & & \big\downarrow{\scriptstyle h_*} & & \\
H^q(\mathfrak{A},\mathfrak{G}') & \xrightarrow{t^{\mathfrak{A}}_{\mathfrak{V}}} & H^q(\mathfrak{V},\mathfrak{G}') & \xrightarrow{t^{\mathfrak{V}}_{\mathfrak{W}}} & H^q(\mathfrak{W},\mathfrak{G}') & \longrightarrow & \cdots & \longrightarrow & H^q(X,\mathfrak{G}')
\end{array}$$

Proof. For any $[f] \in H^q(\mathfrak{A},\mathfrak{G})$ where $f \in Z^q(\mathfrak{A},\mathfrak{G})$, since $h_* t^{\mathfrak{A}}_{\mathfrak{V}}[f] = [h_*\tau^* f]$, $t^{\mathfrak{A}}_{\mathfrak{V}} h_*[f] = [\tau^* h_* f]$ and further,

$$\tau^* h_* f(j_0, j_1, \ldots, j_q) = r^{W'}_{W} h_* f(\tau j_0, \tau j_1, \ldots, \tau j_q) = r^{W'}_{W} h(f(\tau j_0, \tau j_1, \ldots, \tau j_q)),$$
$$h_* \tau^* f(j_0, j_1, \ldots, j_q) = h_* r^{W'}_{W} f(\tau j_0, \tau j_1, \ldots, \tau j_q) = r^{W'}_{W} h(f(\tau j_0, \tau j_1, \ldots, \tau j_q)),$$

where $W' = U_{\tau j_0, \tau j_1, \ldots, \tau j_q}, W = V_{j_0, j_1, \ldots, j_q}$. Compare and get $h_* t^{\mathfrak{A}}_{\mathfrak{V}}[f] = t^{\mathfrak{A}}_{\mathfrak{V}} h_*[f]$. □

Therefore, the long exact sequence can be induced:

$$0 \longrightarrow H^0(X,\mathfrak{G}') \xrightarrow{h'_*} H^0(X,\mathfrak{G}) \xrightarrow{h_*} H^0(X,\mathfrak{G}'') \xrightarrow{\delta^0_*} H^1(X,\mathfrak{G}')$$
$$\cdots \longrightarrow \quad H^q(X,\mathfrak{G}') \xrightarrow{h'_*} H^q(X,\mathfrak{G}) \xrightarrow{h_*} H^q(X,\mathfrak{G}'') \xrightarrow{\delta^q_*} H^{q+1}(X,\mathfrak{G}') \longrightarrow \cdots.$$

In conclusion of the five steps we have the following theorem.

Theorem 3.2. *An exact sequence*

$$0 \longrightarrow \mathfrak{G}' \xrightarrow{h'} \mathfrak{G} \xrightarrow{h} \mathfrak{G}'' \longrightarrow 0$$

of presheaves over the space X gives an long exact cohomology sequence

$$0 \longrightarrow H^0(X,\mathfrak{G}') \xrightarrow{h'_*} H^0(X,\mathfrak{G}) \xrightarrow{h_*} H^0(X,\mathfrak{G}'') \xrightarrow{\delta^0_*} H^1(X,\mathfrak{G}')$$
$$\cdots \longrightarrow \quad H^q(X,\mathfrak{G}') \xrightarrow{h'_*} H^q(X,\mathfrak{G}) \xrightarrow{h_*} H^q(X,\mathfrak{G}'') \xrightarrow{\delta^q_*} H^{q+1}(X,\mathfrak{G}') \longrightarrow \cdots.$$

Theorem 3.2 is also true when the presheaf is replaced by sheaf under the assumption that X is a *paracompact* topological space, in particular a differentiable manifold in our concern. For details, refer to [Hir].

4. De Rham Cohomology of Differentiable Manifolds

The main references of this section are [Griffiths] and [Madsen].

4.1. De Rham Cohomology. Let M be a differentiable manifold. $A^p(M,\mathbb{R})$ is the space of differential p-forms on M. Let d be the exterior differential operator. Let ω be a differential p-form in $A^p(M,\mathbb{R})$. If $d\omega = 0$ then ω is called a *d-closed p-form.* If there exists a $(p-1)$-form η such that $\omega = d\eta$, then ω is called an *d-exact p-form.* Denote

$$Z^p(M,\mathbb{R}) = \{d\text{-closed } p\text{-forms in } A^p(M,\mathbb{R})\};$$
$$B^p(M,\mathbb{R}) = \{d\text{-exact } p\text{-forms in } A^p(M,\mathbb{R})\}.$$

Using the fact that $d^2 = 0$, we can define a quotient group as follows.

DEFINITION 4.1 (De Rham Cohomology Group).

$$H^p_{DR}(M, \mathbb{R}) = \frac{Z^p(M, \mathbb{R})}{B^p(M, \mathbb{R})}$$

is called the *p-th de Rham cohomology group of* M *with coefficients in* $\mathbb{R}$.

Extend the field of coefficients to the complex number field: Let $A^p(M, \mathbb{C})$ denote the space of complex-valued p-forms on M. Let $Z^p(M, \mathbb{C})$ denote the space of complex-valued d-closed p-forms on M and let $B^p(M, \mathbb{C})$ denote the spaces of complex-valued d-exact p-forms on M. Let

$$H^p_{DR}(M, \mathbb{C}) = \frac{Z^p(M, \mathbb{C})}{B^p(M, \mathbb{C})}$$

be the corresponding quotient. This is the *complex p-th de Rham cohomology group*. It can be viewed as the complexified $H^p_{DR}(M, \mathbb{R})$, i.e., $H^p_{DR}(M, \mathbb{C}) = H^p_{DR}(M, \mathbb{R}) \otimes \mathbb{C}$. We will simply state the following fundamental theorem:

THEOREM 4.1 (De Rham Isomorphism). *Let* M *be a compact differentiable manifold, then the de Rham cohomology is isomorphic to the singular cohomology group and also the Čech cohomology group.*

REMARK 4.1. Theorem 4.1 in particular gives: (i) De Rham cohomology is a topological invariant. (ii) If M is a compact oriented n-dimensional differentiable manifold, the Poincaré duality gives

$$H^p_{DR}(M, \mathbb{R}) \cong H_{n-p}(M, \mathbb{Z}), \tag{4}$$

where $H_{n-p}(M, \mathbb{Z})$ is the $(n-p)$-th singular homology group of M with coefficients in $\mathbb{Z}$. (iii) The technique of inducing short exact sequence to long exact sequence in Čech cohomology theory also survives in the de Rham cohomology.

4.2. Homotopic Invariant. We will list several properties of de Rham cohomology and refer the proofs to [Madsen]. A continous map $f : X \longrightarrow Y$ between two topological spaces X and Y is a *homotopy equivalence* if there exists a continuous map $g : Y \longrightarrow X$, such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y. The map g is called a *homotopy inverse* to f. Homotopy equivalence can be shown to be an equivalence relation. Two topological spaces X and Y are said to be *homotopy equivalent* if there exists a homotopy equivalence between them. The equivalence classes of topological spaces defined by the homotopy equivalence are called *homotopy types*.

PROPOSITION 4.2. *De Rham cohomology group is homotopic invariant.*

4.3. Mayer-Vietoris Sequence. This subsection gives a technique, the Mayer-Vietoris sequence, which will be used in the proof of the Poincaré-Hopf theorem in the next section. We will omit the proof which can be found in [Madsen] p. 33. Let U_1 and U_2 be open sets in $\mathbb{R}^n$ and $U = U_1 \cup U_2$. Define $i_\nu : U_\nu \longrightarrow U$ to be the inclusion map where $\nu = 1, 2$. Define $j_\mu : U_1 \cap U_2 \longrightarrow U_\nu$ to be the restriction map where $\nu = 1, 2$.

THEOREM 4.2. *The sequence*

$$0 \longrightarrow \Omega^p(U) \xrightarrow{I^p} \Omega^p(U_1) \oplus \Omega^p(U_2) \xrightarrow{J^p} \Omega^p(U_1 \cap U_2) \longrightarrow 0$$

is exact, where $I^p(\omega) = (i_1^*(\omega), i_2^*(\omega))$ *and* $J^p(\omega_1, \omega_2) = j_1^*(\omega_1) - j_2^*(\omega_2)$.

The exact sequence in the above Theorem 4.2 gives directly the long exact sequence:

DEFINITION 4.2 (Mayer-Vietoris Sequence). The long exact sequence

$$\cdots H^p_{DR}(U) \xrightarrow{I^*} H^p_{DR}(U_1) \oplus H^p_{DR}(U_2) \xrightarrow{J^*} H^p_{DR}(U_1 \cap U_2) \xrightarrow{\partial^*} H^{p+1}_{DR}(U) \cdots ,$$

where $I^*(\omega) = (i_1^*[\omega], i_2^*[\omega])$ and $J^*([\omega_1], [\omega_2]) = j_1^*[\omega_1] - j_2^*[\omega_2]$ is called the *Mayer-Vietoris sequence.*

5. Index and its Equivalence to the Euler Characteristic

Now consider a compact oriented differentiable manifold M of dimension n. As a topological space, it has the Euler characteristic:

$$\chi(M) = \sum_q (-1)^q dim H_q(M, \mathbb{R}). \tag{5}$$

In later context we will show that this sum is always finite whenever M is compact. Hopf in 1937 gives an alternative way to compute the Euler characteristic. By adding a generic smooth vector field X on the manifold M, he obtained

$$\chi(M) = \text{number of zeros of } X\text{, counting multiplicity},$$

in which the right hand side is the notion of *index* of the vector field on M and χ is independent of the vector field X chosen. This new method is remarkable since it gives a differentiable geometrical way to calculate a topological invariant. It can be viewed as the motivation of later works of Todd, Nakano and Hirzebruch. This section is devoted to the proof of this identity. We adopt the proof from [Madsen], Chapter 11 and 12, where it use the de Rham cohomology version of Euler characteristic of the compact oriented differentiable manifold M:

$$\chi(M) = \sum_p (-1)^p dim H^p_{DR}(M, \mathbb{R}).$$

This is equivalent to the definition as in (5). Indeed, since M is oriented, the Poincaré duality and the de Rham isomorphism give the isomorphism in (4). Hence the two definitions agree up to a sign.

5.1. Index of a Vector Field.

5.1.1. *Degree of Smooth Maps between Connected Compact Oriented Differentiable Manifolds of the Same Dimension.* Consider a smooth map $f : N^n \longrightarrow M^n$ between two connected compact oriented differentiable manifolds, a point $P \in M$ is called a *regular value* if the differentiation of f

$$Df_Q : T_Q(N) \longrightarrow T_P(M)$$

is surjective for all $Q \in f^{-1}(P)$. For P being a regular value, we may define the *local index of* f *at* Q for any point $Q \in f^{-1}(P) \subset N$ by

$$Ind(f;Q) = \begin{cases} 1 & detDf_Q > 0 \\ 0 & \text{otherwise.} \end{cases}$$

For the smooth map f we can define the degree of f simply by

$$deg(f) = \sum_{Q \in f^{-1}(P)} Ind(f;Q)$$

where P is any regular value of f. This definition appears to depend on the particular regular value P chosen, but this is not the case. Indeed, we will show that the following commutative diagram

$$\begin{array}{ccc} H^n_{DR}(M,\mathbb{R}) & \xrightarrow{\mathrm{H^n(f)}} & H^n_{DR}(N,\mathbb{R}) \\ {\scriptstyle \int_M}\downarrow & & \downarrow{\scriptstyle \int_N} \\ \mathbb{R} & \xrightarrow{deg(f)} & \mathbb{R}. \end{array}$$

where the map $\int_M : H^n_{DR}(M,\mathbb{R}) \longrightarrow \mathbb{R}$ can be shown to be an isomorphism, (refer to [Madsen] p. 91.) gives

$$\int_N f^*(\omega) = deg(f) \int_M \omega, \tag{6}$$

for any $\omega \in H^n_{DR}(M,\mathbb{R})$. The proof of the identity (6) needs the following lemma whose proof needs the inverse function theorem.

Lemma 5.1. *Let $f : N^n \longrightarrow M^n$ be a smooth map between two compact connected oriented differentiable manifolds. Let $P \in M$ a regular value of f, then the inverse image of $P, f^{-1}(P)$ consists of finitely many points $Q_1, \ldots, Q_k$. Moreover, P has an open neighborhood U in M and each of the $Q_i, i = 1, \ldots, k$ has an open neighborhood V_i in N such that*

(i) *The V_i's are disjoint and $f^{-1}(U) = \cup_{i=1}^k V_i$;*
(ii) *$f_i = f|_{V_i} : V_i \longrightarrow U$ is diffeomorphism for $i = 1, \ldots, k$.*

Theorem 5.1 (Inverse Function Theorem). *Let $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a differentiable map and suppose that at $Q \in U$ the differential*

$$Df_Q : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

is an isomorphism. Then there exists a neighborhood V of Q in U and a neighborhood W of $f(Q) = P$ such that $f : V \longrightarrow W$ is a diffeomorphism.

Proof. (of Lemma 5.1) For any $Q_i \in f^{-1}(P), i = 1, \ldots, k$, since P is regular,

$$Df_{Q_i} : T_{Q_i}N \longrightarrow T_PM$$

is an isomorphism. By the inverse function theorem, f is a diffeomorphism on a neighborhood of Q_i, called W_i. Hence Q_i is the isolated zero in W_i. That is all the Q_i's are isolated from each other and hence we can always choose the W_i's small enough to make any two of them disjoint. Compactness of N implies finiteness of the number of such Q_i's and the V_i's. Let

$$U = \left(\cap_{i=1}^k f(W_i)\right) - f\left(N - \cup_{i=1}^k W_i\right)$$

and define

$$V_i = W_i \cap f^{-1}(U), \quad i = 1, \ldots, k.$$

They are the required neighborhoods. □

Proof. (of the identity (6)) With the same notation in the lemma 5.1,

$$\begin{aligned} deg(f)\int_M \omega &= \sum_{i=1} kInd(f;Q_i)\int_U \omega|_U \\ &= \sum_{i=1}^k \int_{V_i} (f|_{V_i})^*\omega|_U = \int_M f^*(\omega). \end{aligned}$$

□

5.1.2. *Local Index of Singularities of Vector Fields.* With the degree of a smooth map f, we can define the *local index of singularities of a vector field.* Let U be an open set in $\mathbb{R}^n (n \geq 2)$ and $X \in C^\infty(U, \mathbb{R}^n)$ where $C^\infty(U, \mathbb{R})$ is the set of all smooth vector fields on an open set U in a connected compact oriented differentiable manifold M. Assume $0 \in U$ is an isolated zero for X, we call it a *singularity* of the vector field. Fix a sufficiently small positive number ρ such that no other zero lying in the ball centered at 0 and with radius ρ, define a smooth map

$$F_\rho : S^{n-1} \longrightarrow S^{n-1}$$

by $F_\rho(x) = \dfrac{F(\rho x)}{||F(\rho x)||}$. The F_ρ on one hand shrinks the domain of definition of F so that no other singularities are in it and on the other hand normalizes the vector gives a compact target space. F_ρ preserves all the differential property of the original vector field F, so we can define the *local index* of the vector field F at the singularity 0 by the degree of the smooth map F_ρ, that is,

$$\tau(F; 0) = deg(F_\rho).$$

Note that the homotopy class of F_ρ does not depend on the choice of ρ, the local index of F at 0 is thus independent of ρ.

Suppose now X is a smooth vector field on a differentiable manifold M^n, with $P_0 \in M$ be a singularity, i.e. an isolated zero of X. We can also have the notion of local index of X at P_0 by that of the pull back of X on an open set in

$\mathbb{R}^n$: Let $h : U \longrightarrow V \subset \mathbb{R}^n$ be an arbitrary chart around P_0, denoted as (U, h), such that $h(P_0) = 0 \in V$. Then the *local index* $\tau(X, P_0)$ is defined by

$$\tau(X, P_0) = \tau(h_*(X|_U); 0).$$

To show thus defined local index of X is independence of the choices of charts chosen, we need following lemma.

LEMMA 5.2. *If F is a smooth vector field over $W \subset \mathbb{R}^n$ with 0 as an isolated singularity, then for any diffeomorphism $\varphi : W \longrightarrow V$, which maps 0 to 0 and V is an open set containing 0 in $\mathbb{R}^n$. We have $\tau(\varphi_* F; 0) = \tau(F; 0)$.*

Use Lemma 5.2, suppose we have two charts (h, U) and (h', U') around P_0 satisfying $h(P_0) = 0$ and $h'(P_0) = 0$, then define $\varphi : h(U) \longrightarrow h'(U')$ by $\varphi = h' \circ h^{-1}$ then φ will be a diffeomorphism after shrinking U' if necessary and φ maps 0 to 0, hence $\tau(\varphi_*(h_* X); 0) = \tau(h_* X; 0)$ but

$$\tau(\varphi_*(h_* X); 0) = \tau(h'_* \circ h_*^{-1} \circ h_* X; 0) = \tau(h'_* X; 0),$$

hence $\tau(h'_* X; 0) = \tau(h_* X; 0)$. That is the definition of local index of X at P_0 is independent of the charts chosen.

PROOF. (of Lemma 5.2) We may assume that 0 is the only singularity of F on W after shrinking W if necessary. We can find a diffeomorphism $\psi : V \longrightarrow \mathbb{R}^n$ after shrinking V if necessary maps 0 to 0. Then if we prove the assertion for ψ and $\psi \circ \varphi$ that is both $\tau(\psi_*(\varphi_* F); 0) = \tau(\varphi_* F; 0)$ and $\tau((\varphi \circ \psi)_* F; 0) = \tau(F; 0)$, then since $\psi_*(\varphi_* F) = (\psi \circ \varphi)_* F$ we will have $\tau(\varphi_* F; 0) = \tau(F; 0)$. Hence it reduces to prove the following statement: Let $\varphi : U \longrightarrow \mathbb{R}^n$ be a diffeomorphism, where $Y = \varphi_* F \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ has the origin as its only singularity and φ maps 0 to 0, then $\tau(\varphi_* F; 0) = \tau(F; 0)$. Let $U_0 \subset U$ be a star-shaped open neighborhood of 0, we define a homotopy $\Psi : U_0 \times [0, 1] \longrightarrow \mathbb{R}^n$ by

$$\Psi_t(x) = \Psi(x, t) = \begin{cases} (D\varphi)x & \text{if } t = 0 \\ \frac{\varphi(tx)}{t} & \text{if } t \neq 0. \end{cases}$$

For $x \in U_0$,

$$\begin{aligned} \varphi(x) &= \varphi(1 \cdot x) - \varphi(0 \cdot x) \\ &= \int_0^1 \frac{d}{dt}\varphi(tx)dt \\ &= \int_0^1 \left(\sum_{i=1}^n x_i \frac{\partial \varphi}{\partial x_i}(tx) \right) dt \\ &= \sum_{i=1}^n x_i \int_0^1 \frac{\partial \varphi}{\partial x_i}(tx)dt = \sum_{i=1}^n x_i \varphi_i(x), \end{aligned}$$

where $\varphi_i(x) = \int_0^1 \frac{\partial \varphi}{\partial x_i}(tx)dt$. Note that $\varphi_i \in C^\infty(U_0, \mathbb{R}^n)$. Hence for $t \neq 0$,

$$\Psi(x, t) = \frac{\varphi(tx)}{t} = \sum_{i=1}^n (tx_i) \frac{\varphi_i(tx)}{t} = \sum_{i=1}^n x_i \varphi_i(tx)$$

by the continuity of the homotopy Ψ we have when $t = 0$, $\Psi(x, 0) = \sum_{i=1}^{n} x_i\varphi(0)$. With this expression of Ψ, on $U_i \times [0, 1]$ we can use the same formula to extend the domain of definition of Ψ to an open set $\tilde{U}$ where $U_o \times [0, 1] \subset \tilde{U} \subset U_o \times \mathbb{R}$. Start from the vector field $Y = \varphi_* F \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, it can be restricted on $\Psi_t(U_0)$ for any fixed $t \in [0, 1]$. Since Ψ_t is a diffeomorphism from U_0 to an open set of $\mathbb{R}^n$, hence we may push forward Y by ${\Psi_t^{-1}}_*$ to get the vector field

$$X_t = {\Psi_t^{-1}}_*(Y) \in C^\infty(U_0, \mathbb{R}^n).$$

Note the $X_t(x)$ is smooth on $\tilde{U}$. In particular, consider the case when $t = 0$ and $t = 1$: $X_0 = (\Psi_0^{-1})_* Y = {(D\varphi)_0^{-1}}_* Y$ and $X_1 = {\Psi_1^{-1}}_*(Y) = {\varphi^{-1}}_* Y = F|_{U_0}$. Choose $\rho > 0$ such that $D(0; \rho) \subset U_0$, define a homotopy $G : S^{n-1} \times [0, 1] \longrightarrow S^{n-1}$ by

$$G(t, x) = \frac{X_t(\rho x)}{||X_t(\rho x)||}, \quad t \in [0, 1],$$

which gives $G(0, x) = \dfrac{X_0(\rho x)}{||X_0(\rho x)||}$ and $G(1, x) = \dfrac{X_1(\rho x)}{||X_1(\rho x)||}$. That is the smooth map $F_0 : S^{n-1} \longrightarrow S^{n-1}$ defined by $F_0(x) = \dfrac{X_0(\rho x)}{||X_0(\rho x)||} \in S^{n-1}$ and the smooth map $F_1 : S^{n-1} \longrightarrow S^{n-1}$ defined by $F_1(x) = \dfrac{X_1(\rho x)}{||X_1(\rho x)||} \in S^{n-1}$ are homotopic. By the fact that index of a smooth map is homotopic invariant, we have $\tau(F_0; 0) = \tau(F_1; 0)$ by definition of index of vector field on open set U containing 0 and with 0 as the singularity, we immediately have $\tau(X_0; 0) = \tau(F_0; 0)$ and $\tau(X_1; 0) = \tau(F_1; 0)$, hence $\tau(X_0; 0) = \tau(X_1; 0)$. That is $\tau({(D\varphi)_0^{-1}}_* Y; 0) = \tau(F; 0)$. So it remains to show that $\tau({(D\varphi_0^{-1}}_* Y; 0) = \tau(Y; 0)$. To proceed, we need a fact: let $Z \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ be a smooth vector field with the origin as the only singularity. Then the mapping

$$\times\tau(Z; 0) : H^{n-1}(\mathbb{R}^n - \{0\}) \longrightarrow \mathbb{R}$$

is an isomorphism. Proof of the fact: Let $i : S^{n-1} \longrightarrow \mathbb{R}^n - \{0\}$ be the inclusion and $r : \mathbb{R}^n - \{0\} \longrightarrow S^{n-1}$ the map $r(x) = \dfrac{x}{||x||}$. Define a smooth map $Z_1 : S^{n-1} \longrightarrow S^{n-1}$ by $Z_1 = r \circ Z \circ i$. Then by definition of the local index of Z, we have $\tau(Z; 0) = deg(Z) = deg(Z_1)$. Since the following diagram is commute so the fact follows.

$$\begin{array}{ccc} H^{n-1}(\mathbb{R}^n - \{0\}) & \xrightarrow{\mathrm{H^{n-1}(Z)}} & H^{n-1}(\mathbb{R}^n - \{0\}) \\ \Big\downarrow {\scriptstyle \mathrm{H^{n-1}(r)}} & & \Big\downarrow {\scriptstyle \mathrm{H^{n-1}(i)}} \\ H^{n-1}(S^{n-1}) & \xrightarrow{\mathrm{H^{n-1}(Z_1)}} & H^{n-1}(S^{n-1}), \end{array}$$

in which i and r are inverse isomorphisms. Apply the fact both on $Z = F$ and on $Z = (D{\varphi_0^{-1}}_* Y)$. If $F_1 = r \circ F \circ i = r \circ {\phi^{-1}}_* Y \circ i$, then $\tau(F; 0) = deg(F_1)$. If $G_1 = r \circ D{\varphi_0^{-1}}_* Y \circ i$, then $\tau(D{\varphi_0^{-1}}_* Y; 0) = deg(G_1; 0)$. Since $G_1 = F_1$, $\tau(F; 0) = \tau(D{\varphi_0^{-1}}_* Y; 0)$. $\square$

5.1.3. *Total Index of a Vector Field.*

Definition 5.1 (Total Index). Let M be a differentiable manifold and R be a compact set in M, let X be a smooth vector field on M with only isolated singularities then the total index of X over R is defined by

$$Index(X;R) = \sum \tau(X;P)$$

where the summation runs through all the singularities P of X lies in R. If the differentiable manifold M itself is compact, we can define the total index of the smooth vector field X on M by

$$Index(X) = Index(X;M).$$

5.2. Poincaré-Hopf Theorem. With the definition of the (total) index of a smooth vector field over a differentiable manifold, we will in this subsection show the important Poincaré-Hopf theorem, which gives the calculation of the Euler characteristics by the total index of a vector field on a compact oriented differentiable manifold. We will first show that any smooth vector field over a compact oriented differentiable manifold will give the same total index. Hence when we compute the total index of a compact oriented differentiable manifold we are free to choose a particular good one which is easy to compute. On the other hand when calculating the Euler characteristic of a compact oriented differentiable manifold, we may decompose the manifold to fragments in such a way that we know how to compute the Euler characteristics. Both sides can be achieved by the so-called *Morse function.* That is we can choose a gradient-like vector field of a Morse function when computing the index of vector field and we can also use the same Morse function to decompose our compact oriented manifold into fragments. This is the idea of proof of the Poincaré-Hopf theorem from [Madsen].

5.2.1. *Total Index is a Topological Invariant.* Now we will show the total index of vector field of a compact oriented manifold is independent of the vector fields chosen. In other words, the total index of vector field is a topological invariant of a compact oriented differentiable manifold. We will first give the notion of non-degenerate vector fields and show that the total index of vector fields is equal to the total index of the non-degenerate ones, then show that all non-degenerate vector fields will give the same total index.

Let $M^n \subset \mathbb{R}^{n+k}$ be a compact oriented differentiable manifold, at each point $P \in M$ there is a normal vector space $T_PM^\perp$ defined by $T_P\mathbb{R}^{n+k}/T_PM^n$ of dimension k. Let W be an open set in M a smooth map $G : W \longrightarrow \mathbb{R}^{n+k}$ is called a *Gauss map* on W if $G(P) \in T_PM^\perp$ and $G(P)$ points outwards and is of unit length for all $P \in W$.

Lemma 5.3. *Let $F \in C^\infty(U, \mathbb{R}^n)$ be a vector field on an open set $U \subset \mathbb{R}^n$, with only isolated zeros. If $R \subset U$ is an compact domain with smooth boundary ∂R, and $F|_{\partial R}$ is never zero. Then if $f : \partial R \longrightarrow S^{n-1}$ is defined by $f(x) = \dfrac{F(x)}{||F(x)||}$, then*

$$Index(F;R) = deg(f).$$

This implies that $Index(F;R)$ depends only on the restriction of F to ∂R.

PROOF. Since R is compact we may assume $P_1, \dots, P_k$ where k is finite, be the zeros of F in R. Choose disjoint closed balls $D_j \subset R - \partial R$ with center P_j for $j = 1, \dots, k$. For each such j define $f_j : \partial D_j \longrightarrow S^{n-1}$ by $f_j(x) = \dfrac{F(x)}{||F(x)||}$. Denote the compact set $R - \cup_j D_j^0$ by X. Hence $\partial X = \partial R \cup \partial D_1 \cup \cdots \cup \partial D_k$. Define $g : \partial X \longrightarrow S^{n-1}$ by

$$g(x) = \begin{cases} f(x) & x \in \partial R \\ f_i(x) & x \in \partial D_i \end{cases}.$$

Hence $deg(g) = -\sum_{j=1}^{k} deg(f_j) + deg(f)$. However, for any $(n-1)$-form ω on S^{n-1} such that $\int_{S^{n-1}} \omega = 1$, we have

$$deg(g) = \int_{\partial X} g^*\omega = \int_X dg^*(\omega) = 0.$$

Hence,

$$deg(f) = \sum_{j=1}^{k} deg(f_j).$$

By definition of local index, $deg(f_j) = \tau(F; P_j)$, we have

$$Index(F;R) = \sum_{j=1}^{k} \tau(F;P_j) = \sum_{j=1}^{k} deg(f_j) = deg(f).$$

□

Let $G : \partial R \longrightarrow S^{n-1}$ be a Gauss map. Then G is homotopic to the f in Lemma 5.3 by the homotopy $\dfrac{(1-t)f(p) + tG(p)}{||(1-t)f(p) + tG(p)||}, t \in [0,1]$. By the fact that degree of a smooth map is a homotopic invariant, we have $deg(g) = deg(f)$. Therefore, we have:

COROLLARY 5.4. *Let $F \in C^\infty(U, \mathbb{R}^n)$ be a vector field on an open set $U \subset \mathbb{R}^n$, with only isolated zeros. If $R \subset U$ is an compact domain with smooth boundary ∂R, and $F|_{\partial R}$ is never zero. Let $G : \partial R \longrightarrow S^{n-1}$ be the Gauss map. Then,*

$$Index(F;R) = deg(G). \tag{7}$$

Before going on, we introduce the definition of non-degenerate singularity. Let X be a smooth vector field on a differentiable manifold and let $P_0 \in M$ be a zero, if (U, h) is a coordinate chart around P_0 and h maps P_0 to $0 \in \mathbb{R}^n$, then let $F = h_*(X|_U) \in C^\infty(h(U), \mathbb{R}^n)$ the induced vector field on a neighborhood of 0 in $\mathbb{R}^n$. If the differential of F at 0, $DF_0 : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is an isomorphism, then we call P_0 is a *non-degenerate singularity.* Note that if P_0 is a non-degenerate singularity, then by the inverse function theorem, F is a local diffeomorphism around 0, which implies that 0 is an isolated zero for F hence P_0 is also an isolated zero for X.

LEMMA 5.5. *Suppose $F \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ has the origin as its only zero. Then there exists an $\tilde{F} \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$, with only non-degenerate zeros, that coincides with F outside a compact set.*

PROOF. The proof is by construction: Let $\phi \in C^\infty(\mathbb{R}^n, [0,1])$ be a smooth function such that

$$\phi(x) = \begin{cases} 1 & \text{if } ||x|| \leq 1; \\ 0 & \text{if } ||x|| \geq 2. \end{cases}$$

Define

$$\tilde{F}(x) = F(x) - \phi(x)\omega$$

where $\omega \in \mathbb{R}^n$ is chosen such that:

$$||\omega|| < \inf_{1 \leq ||x|| \leq 2} ||F(x)|| = c.$$

Hence for $1 \leq ||x|| \leq 2$, $||\tilde{F}(x)|| \geq c \geq c - ||w|| > 0$. Hence all the zeros of $\tilde{F}$ lie in the open unit ball D^n. Since $\tilde{F}|_{D^n} = F - \omega|_{D^n}$, then $\tilde{F}^{-1} = D^n \cap F^{-1}(\omega)$. We can always chose an ω which is a regular value, by a fact from analysis saying that the set of regular value is dense in the image set. Then for all $p \in \tilde{F}^{-1}(0)$, $D\tilde{F}_p = DF_p$ and is invertible. Further notice that $\tilde{F}(x) = F(x)$ for $||x|| > 2$, thus constructed $\tilde{F}$ has the desired properties. □

THEOREM 5.2. *Let X be a smooth vector field on the compact oriented differentiable manifold M^n with only isolated singularities. Then there exists a smooth vector field $\tilde{X}$ on M having only non-degenerate singularities such that*

$$Index(X) = Index(\tilde{X}).$$

PROOF. For each zero of X we may choose disjoint coordinate charts around it. Apply Lemma 5.5 on each charts, all the induced vector fields can be piece together nicely. The resulting vector field $\tilde{X}$ on M is with only non-degenerate zeros. Summing all the local indexes, we finally have $Index(X) = Index(\tilde{X})$. □

Thus, to show that the total index of a vector field over a compact oriented differentiable manifold is independent of vector fields chosen, it suffices to show the statement for the vector fields with only non-degenerate zeros.

THEOREM 5.3. *Let M^n be a compact oriented differentiable manifold in $\mathbb{R}^{n+k}$, and let X be a smooth vector field over M with only isolated zeros. Then $Index(X)$ is independent of X.*

We will outline the idea of proof first: By Theorem 5.2, we may, without loss of generality, assume that X has only non-degenerate zeros. As the result given by Corollary 5.4, the total index of a vector field on a compact domain is the same as the degree of Gauss map from the boundary of the compact domain to the unit sphere. The whole proof will be the same as that of Corollary 5.4 if we can regard our compact oriented manifold M^n as a compact domain of some larger differentiable manifold, say $\tilde{M}$, and if we can extend the vector field X

smoothly on to the larger manifold, say $\tilde{X}$. Then the Corollary 5.4 will give the total index of $\tilde{X}$ on the compact domain M be the degree of Gauss map from ∂M to some unit sphere, under the two conditions that (i) ∂M is again smooth and (ii) The restriction of $\tilde{X}$ on ∂M is never zero. But the last two conditions may not generally satisfied by an arbitrary compact oriented manifold M and a smooth vector field X on it. We will then instead of consider M itself, consider a better shaped manifold, namely the tubular neighborhood N_ε of M and also extend X to the tubular neighborhood, call it X_ε. This kind of extension is nice since we can show that the two pairs $(N_\varepsilon, X_\varepsilon)$ and (M, X) have the same total index. Further, the tubular neighborhood and the extended vector field $(N_\varepsilon, X_\varepsilon)$ do satisfy the two conditions and give the answer. The detailed proof is as follows:

Proof. We will construct a *tubular neighborhood* N_ε of M first. Choose an arbitrary geodesic $c(t) : t \in [-a, a]$ for some $a > 0$ through the whole compact oriented manifold M. Assign each point on the geodesic a unit disk $D(t)^{n+k-1}(\varepsilon)$ centered at $c(t)$ with radius $\varepsilon > 0$ and is lying $V(t)^{n+k-1}$, the vector space transversal to the vector $c'(t)$. Since M is compact we can always choose a sufficiently large radius ε such that at each point $c(t)$ the intersection $M \cap V(t)^{n+k-1}$ lies in the unit disk $D(t)^{n+k-1}(\varepsilon)$. The union of the disks thus defined $N_\varepsilon = \cup_{t\in[-a,a]} D(t)^{n+k-1}(\varepsilon)$ is the tubular neighborhood of radius ε around M. Note that now the boundary ∂N_ε is smooth. This is the condition (i). Now we are going to extend the vector field X on M to a vector field X_ε on N_ε in a nice way such that X_ε when restricted on ∂N_ε gives zero. : Change the ε to $\varepsilon + \Delta\varepsilon$ for some small positive number $\Delta\varepsilon$ in the N_ε, we get a larger tubular neighborhood $N = N_{\varepsilon+\Delta\varepsilon}$, Define $X_\varepsilon \in C^\infty(N, \mathbb{R}^n)$ by

$$X_\varepsilon(q) = X(\pi(q)) + (q - \pi(q)).$$

Note that the restriction of the vector field X_ε on N_ε points outwards. In fact, For $Q \in \partial N_\varepsilon, Q - \pi(Q) \perp T_Q \partial N_\varepsilon$ by the construction of the tubular neighborhood. Since $Q - \Pi(Q) \perp X(\pi(Q))$ (X is tangent to M) and $T_Q(\partial N_\varepsilon)$ is of dimension $n + k - 1$, hence $X(\pi(Q)) \in T_Q \partial N_\varepsilon$, which implies that X_ε is pointing outwards. Observe that zeros of X_ε on N are zeros of X on M. Indeed, the two components $X(\pi(Q))$ and $(Q - \pi(Q))$ are orthogonal gives

$$X_\varepsilon(Q) = 0 \Longleftrightarrow \begin{cases} X(\pi(Q)) = 0 \\ Q = \pi(Q) \end{cases} \Longleftrightarrow \begin{cases} X(Q) = 0 \\ \pi(Q) = Q \end{cases} .$$

Thus, $X_\varepsilon(P) \neq 0$ for all $P \in \partial N_\varepsilon$. This is the condition (ii). Therefore we have $Index(X_\varepsilon; N_\varepsilon) = deg(g)$ where $g : \partial N_\varepsilon \longrightarrow S^{n+k-1}$ stands for the Gauss map pointing outwards.

It remains to show that $Index(X_\varepsilon; N_\varepsilon) = Index(X; M)$. It suffices to show that for each zero P of X on M, $\tau(X; P) = \tau(X_\varepsilon; P)$. In local coordinates around any zero P of X in M with P corresponds to $0 \in \mathbb{R}^n$. The vector field

X can be written in the form

$$X = \sum_{i=1}^{n} f_i(x) \frac{\partial}{\partial x_i}$$

with $f_i(0) = 0, i = 1, \ldots, n$. With these coordinates we have

$$\tau(X; P) = \text{ sign of } det \left(\frac{\partial f_i}{\partial x_j}(0) \right).$$

On the other hand, since $X_\varepsilon(Q) = X(\pi(Q) + (Q - \pi(Q))$, the differential $DX_{\varepsilon P} : \mathbb{R}^{n+k} \longrightarrow \mathbb{R}^{n+k}$ is identity map when restricted on $T_P M^\perp$. Hence,

$$DX_{\varepsilon P} : T_P M \longrightarrow T_P M$$

is associated with the same matrix $\left(\frac{\partial f_i}{\partial x_j}(0) \right)$ as that of X. This implies that the Jacobian of the vector field X_ε at P has the same sign as the Jacobian of X at P and further P is also a non-degenerate zero for the vector field X_ε. Hence $\tau(X; P) = \tau(X_\varepsilon; P)$ and then $Index(X; M) = Index(X_\varepsilon; N_\varepsilon)$. Finally, we have

$$Index(X; M) = deg(G).$$

In particular, $Index(X)$ is independent of the particular choice of X. □

5.2.2. *Morse Function.* To show that total index is equal to the Euler characteristics, it suffices to show their agreement in the case given by the Morse function. Let us introduce the Morse function first.

Definition 5.2 (Morse Function). Let M be a differentiable manifold and let $f \in C^\infty(M, \mathbb{R})$ be a function on M, a critical point $P \in M$ of the function f is said to be *non-degenerate* if the Hessian of f at P is invertible. We call f a *Morse function* if all critical points of f are non-degenerate.

Note that the invertibility of a Hessian is independent of particular choices of coordinates it use. We further define the *index* of a non-degenerate critical point p by the maximal dimension of a subspace $V \subset T_P M$ for which the restriction of $D^2 f_P$ is negative definite. For example, let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a function given by

$$f(x) = c - x_1^2 - x_2^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$$

for some $\lambda, 0 \leq \lambda \leq n$ and $c \in \mathbb{R}$. The gradient of f at $x \in \mathbb{R}^n$ is given by

$$grad_x(f) = 2(-x_1, \ldots, -x_\lambda, x_{\lambda+1}, \ldots, x_n)$$

hence 0 is the only critical point of f and further the Hessian of f at 0 is given by

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(0) \right) = diag(-2, \ldots, -2, 2, \ldots, 2)$$

the diagonal matrix with λ terms of -2 and $(n - \lambda)$ terms of 2. Hence the index of 0 is λ and further the vector field $grad(f)$ is of total index equals to the local index at 0, given by $(-1)^\lambda$.

The fact is that for any smooth function $g \in C^\infty(M, \mathbb{R})$, there exists a chart (h, U) around P and maps P to $0 \in \mathbb{R}^n$ such that the function $g \circ h^{-1}(x)$ is in the form of the function as above, $f(x) = c - x_1^2 - x_2^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$. The proof is in [Madsen] p. 117.

DEFINITION 5.3 (Gradient-like vector fields). Let $f \in C^\infty(M, \mathbb{R})$ be a Morse function. A smooth vector field X on M is said to be *gradient-like* for f, if the following is true:

(i) For every non-critical point $P \in M, Df_P(X(P)) > 0$.
(ii) If $P \in M^n$ is a critical point of f, then there exists a chart (h, U) around P such that $h(P) = 0 \in \mathbb{R}^n$ and further $f \circ h^{-1}(x) = f(P) - x_1^2 - x_2^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$, and $h_* X|_U = grad(f \circ h^{-1})$.

There is also a fact that every smooth function has a gradient-like vector field of it. The proof is in [Madsen] p.119.

5.2.3. *Poincaré-Hopf Theorem.* With the notion of Morse function and its gradient-like vector field, we are in the position to calculate the total index and the Euler characteristics in this case and prove the following theorem.

THEOREM 5.4 (Poincaré-Hopf Theorem). *Let X be a smooth vector field on a compact oriented differentiable manifold M^n, with only isolated singularities. Then*

$$\chi(M) = Index(X),$$

where $\chi(M) = \sum_i (-1)^p dim H^p_{DR}(M, \mathbb{R})$ is the Euler characteristic.

PROOF. It suffices to compute one particular case and show the resulting total index and the Euler characteristics are of the same value. Let M be the compact oriented differentiable manifold and let f be a Morse function on it, and X be the gradient-like vector field of f. Any critical point $P \in M$ of the function f is non-degenerate by definition of Morse function. Assume P has index λ. Since there is a coordinate chart (U, h) which maps P to $0 \in \mathbb{R}^N$ and in the form that

$$f \circ h^{-1}(x) = f(P) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2, \quad x \in h(U).$$

In this case, by definition of local index of a vector field,

$$\tau(X; P) = \tau(h_* X; 0) = (-1)^\lambda$$

Hence if c_λ is the number of critical points of f with index λ, then

$$Index(X) = \sum_{\lambda=1}^{n} (-1)^\lambda c_\lambda. \tag{8}$$

Now we will use the same Morse function f to compute the Euler characteristics of M. The method is that we use the function to decompose the manifold in such a way that Euler characteristics of the components, which are related to spheres and hence easy to compute. For any $a \in \mathbb{R}$, let

$$M(a) = \{P \in M | f(P) < a\}.$$

There are several facts from dynamical systems and ordinary differential equations which we will use. The proofs can be found in [Madsen], Appendix C.

LEMMA 5.6. *If there are no critical value in the interval* $[a_1, a_2]$, *then* $M(a_1)$ *and* $M(a_2)$ *are diffeomorphic.*

LEMMA 5.7. *Suppose that* a *is a critical value and* $P_1, \ldots, P_r$ *are the corresponding critical points in* $f^{-1}(a)$. *Let* λ_i *denote the index of* P_i *for* $i = 1, \ldots, r$. *Then there exists an* $\varepsilon > 0$ *and disjoint open neighborhoods* U_i *of* P_i *for each* $i = 1, \ldots, r$, *such that*

(i) $P_1, \ldots, P_r$ *are the only critical points in* $f^{-1}([a - \varepsilon, a + \varepsilon])$.
(ii) U_i *is diffeomorphic to an open contractible subset of* $\mathbb{R}^n$.
(iii) *Then the intersection* $U_i \cap M(a - \varepsilon)$ *is diffeomorphic to* $S^{\lambda_i - 1} \times V_i$, *where* V_i *is an open contractible subset of* $\mathbb{R}^{n - \lambda_i + 1}$.
(iv) $M(a + \varepsilon)$ *is diffeomorphic to* $U_1 \cup \cdots \cup U_r \cup M(a - \varepsilon)$.

Choose an ε as in Lemma 5.7, we will show first the identity:

$$\chi(M(a + \varepsilon)) = \chi(M(a - \varepsilon)) + \sum_{i=1}^{r} (-1)^{\lambda_i}. \tag{9}$$

Since each $i = 1, \ldots, r$, U_i is an open contractible set in $\mathbb{R}^n$, then by the fact that

$$H^p_{DR}(U_i, \mathbb{R}) = \begin{cases} 0 & \text{when } p > 0 \\ \mathbb{R} & \text{when } p = 0. \end{cases}$$

We have for the disjoint union $U = U_1 \cup \cdots \cup U_r$,

$$H^p_{DR}(U, \mathbb{R}) = \begin{cases} 0 & \text{when } p > 0 \\ \mathbb{R}^r & \text{when } p = 0. \end{cases}$$

Hence

$$\chi(U) = \sum (-1)^p dim H^p_{DR}(U, \mathbb{R}) = dim H^0_{DR}(U, \mathbb{R}) = r.$$

Since for each $i = 1, \ldots, r$, $U_i \cap M(a - \varepsilon)$ is homotopic to $S^{\lambda_i - 1}$ and (iii) of Lemma 5.7,

$$\chi(U_i \cap M(a - \varepsilon)) = \chi(S^{\lambda_i - 1}) = 1 + (-1)^{\lambda_i - 1}$$

where the second equality is by the fact that

$$H^p(S^n; \mathbb{R}) = \begin{cases} \mathbb{R} & \text{when } p = 0 \text{ or } n \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\chi(U \cap M(a - \varepsilon)) = \sum_{i=1}^{r} \chi(U_i \cap M(a - \varepsilon)) = \sum_{i=1}^{r} 1 + (-1)^{\lambda_i - 1} = r - \sum_{i=1}^{r} (-1)^{\lambda_i}.$$

Apply the formula which is by Mayer-Vietoris sequence in Definition 4.2:

$$\chi(U \cup V) = \chi(U) + \chi(V) - \chi(U \cap V)$$

by substituting $V = M(a-\varepsilon)$, we have

$$\begin{aligned} \chi(U \cup M(a-\varepsilon)) &= \chi(U) + \chi(M(a-\varepsilon)) - \chi(U \cap M(a-\varepsilon)) \\ &= r + \chi(M(a-\varepsilon)) - r + \sum_{i=1}^{r} (-1)^{\lambda_i} \\ &= \chi(M(a-\varepsilon)) + \sum_{i=1}^{r} (-1)^{\lambda_i}. \end{aligned}$$

Since $U \cup M(a-\varepsilon)$ is diffeomorphic to $M(a+\varepsilon)$ and (iii) of Lemma 5.7, then

$$\chi(U \cup M(a-\varepsilon)) = \chi(M(a+\varepsilon))$$

and hence the identity (9).

Let $a_1 < a_2 < \cdots < a_{k-1} < a_k$ be all the critical values of f on M. Choose $b_0, \ldots, b_k$ by

$$\begin{cases} b_0 < a_1 \\ b_j \in (a_j, a_{j+1}) \quad \text{for } j = 1, \ldots, k-1 \\ b_k > a_k. \end{cases}$$

Then by Lemma 5.6, $dimH^p(M(b_j))$ is independent of choices of b_j. By the identity (9), we have

$$\chi(M(b_j)) - \chi(M(b_{j-1})) = \sum_{P \in f^{-1}(a_j)} (-1)^{\lambda(P)}, \quad j = 1, \ldots, k,$$

where the summation runs over all critical points of a_j, and $\lambda(P)$ is the index of the critical points. Hence sum the identities for all $j = 1, \ldots, k$. After cancellation, we have

$$\chi(M) = \sum (M(b_k)) = \sum_{P} (-1)^{\lambda(P)}$$

where the summation runs over all the critical points of f. If we denote c_λ by the number of critical points with index λ, then it is equivalent to write down

$$\chi(M) = \sum_{\lambda=0}^{n} (-1)^\lambda c_\lambda.$$

Compare this identity with identity (8), we finally have $Index(X) = \chi(M)$. $\square$

CHAPTER 2

Vector Bundles and Characteristic Classes

In this chapter, we will first give the notion of fiber bundles, including principal bundles and vector bundles. After introducing metrics of vector bundles, we will introduce connections and curvatures of vector bundles. The main references are [Hir], [Mok], [Griffiths] and [Kobayashi]. Motivated by the Hopf-Poincaré theorem, there is a generalized differential geometrical way to measure topological invariants of vector bundles. Characteristic classes are important object appearing in this generalization. The rest of this chapter will concentrate on characteristic classes. We will sketch the way of defining the *Stiefel-Whitney classes* and hence the *Chern classes* in terms of Schubert cycles from [Chern2] Appendix. Other references are [Chern1], [Bor-Hir], [Milnor], [Hir] and [Griffiths]. We will also give the way from [Hir] to define Chern classes by Čech cohomology in which almost the same strategy is used as the original definition. As pointed out by S. S. Chern, the characteristic classes are topological invariants which measure the difference from a vector bundle to a trivial product structure. Since the measurement can also be represented by curvature, he gave a curvature representation of Chern classes. For convenience, we will give an axiomatic way of defining Chern classes from [Hir].

1. Fiber Bundles and Vector Bundles

Let M and F be topological spaces and G be a Lie group acting from left effectively on F. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of M. Consider the disjoint union $\coprod_{i\in I} U_i \times F$. They can be pieced together nicely, if points in the intersection of open sets in the covering have unambiguous correspondence. Let a point $x \in U_i \cap U_j$ for certain $i, j \in I$. The objective is to identify a certain pair $(x, v_i) \in (U_i \times F)$ with another pair $(x, v_j) \in (U_j \times F)$ in a consistent way. Let

$$g_{ij} : U_i \cap U_j \longrightarrow G,$$

or equivalently,

$$g_{ij}(x) : F^{(i)} \longrightarrow F^{(j)}$$

where $g_{ij} \in \Gamma(U_i \cap U_j, G_c)$ with G_c being the sheaf of germs of continuous maps from $U_i \cap U_j$ to G and $F^{(i)}$ is denoted F as a subset of $U_i \times F$ for any $i \in I$. We may assume $g_{ij}(\varnothing) = 0$. The map induces an equivalence relation between elements $(x, v_i) \in (U_i, F)$ and $(x, v_j) \in (U_j, F)$ if and only if it satisfies the following conditions: For any $x \in M, v_i \in F^{(i)}$ for any $i \in I$,

(i) $g_{ij}(x)(v_i) = v_j$ if and only if $g_{ji}(x)(v_j) = v_i$,

(ii) $g_{ij}(x)(0) = 0$,
(iii) $g_{ij}(x)(v_i) = v_j$, $g_{jw}(x)(v_j) = v_w$ implies $g_{iw}(x)(v_i) = v_w$.

Denote the equivalence relation by $\sim$. That is, $(x, v_i) \sim (x, v_j)$ if and only if $g_{ij}(x)(v_i) = v_j$ for any $x \in M$. Define W to be the quotient space $\coprod_{i \in I} U_i \times F / \sim$. Then the identification space W with the projection $\pi : W \longrightarrow M$ which project (x, v_i) to $x \in M$ is called a *fiber bundle over M with fiber F and structure group G*. The $\{g_{ij}\} \subset \Gamma(U_i \cap U_j, G_c)$ are called the *transition functions* of the fiber bundle.

If M is a complex manifold, F is $\mathbb{C}^q$, the group G is $GL(q, \mathbb{C})$ and the transition functions $g_{ij} : U_i \cap U_j \longrightarrow GL(q, \mathbb{C})$ are in $\Gamma(U_i \cap U_j, G_b)$ where G_b is the sheaf of germs of differentiable functions from $U_i \cap U_j$ to G. Then the identification space W is called a *vector bundle.* Furthermore, if all the $g_{ij} \in \Gamma(U_i \cap U_j, G_\omega)$ where G_ω denotes the sheaf of germs of holomorphic functions from $U_i \cap U_j$ to G, then W is called a *holomorphic vector bundle.* When the fiber F equals to the structure group G and adopts the convention of right action instead of left action, then the resulting fiber bundle is called a *principal bundle.* In a less intuitive but more standard way, the definition of fiber bundle can be reformulated as follows,

Definition 1.1 (Fiber Bundle). A topological space W, a complex manifold M, together with a projection map $\pi : W \longrightarrow M$ is called a *fiber bundle* with fiber F and structure group G if there exists a system of coordinate transformations:

(i) an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of M and a homeomorphism
$$h_i : \pi^{-1}(U_i) \longrightarrow U_i \times F$$
which maps the fiber $\pi^{-1}(x)$ on to $x \times F$,
(ii) elements $g_{ij} \in \Gamma(U_i \cap U_j, G_c)$ for all $i, j \in I$ such that
$$h_i \circ h_j^{-1}(x, v_j) = (x, g_{ij}(x)v_j)$$
for all $x \in U_i \cap U_j, v_j \in F^{(j)}$.

The mapping h_i for $i \in I$ is called a *local trivialization* or a *chart* of the fiber bundle.

Remark 1.1. This way of definition can be viewed as a reversed way of the first way since it starts from the total space W and then to separate it to pieces which are trivial product structure $U_i \times F$ by the operation of h_i. The composition $h_i \circ h_j^{-1}$ evaluated at x gives a correspondence between elements in $F^{(i)}$ and those in $F^{(j)}$. However, the well-definedness of the function h_i again gives the requirement that $\{g_{ij}\}$ will give an equivalence relation.

For a fixed $k \in I$, a chart $h_k : \pi^{-1}(U_k) \longrightarrow U \times F$ is called an *admissible chart* if there are elements $g_{ik} \in \Gamma(U_i \cap U_k, G_c)$ for each $i \in I$ such that
$$h_k \circ h_i^{-1}(x, v_i) = (x, g_{ki}(x)v_i)$$

For all $x \in U_k \cap U_i$ and $v_i \in F^{(i)}$. Two systems of coordinate transformations define the same fiber bundle if and only if every admissible chart for one system is an admissible chart for the other system. Let $u \in U_i \cap U_j$, then $h_i \circ h_j^{-1}$: $U_i \cap U_j \times F \longrightarrow U_i \cap U_j \times F$ such that $h_i \circ h_j^{-1}(x, v_j) = (x, g_{ij}(v_j))$ where g_{ij} is a section in $\Gamma(U_i \cap U_j, G_c)$. It can be shown easily that the $\{g_{ij}\}$ satisfy the cocycle conditions and hence a cocycle in $Z^1(\mathcal{U}, G_c)$. Conversely, given an element $\{g_{ij}\}$ in $Z^1(\mathcal{U}, G_c)$, we can use them as transition functions and determine a fiber bundle W.

Instead of considering cocycles with respect to a certain covering, we may consider a class in $H^1(M, G_c)$ represented by a certain cocycle $\{g_{ij}\} \in Z^1(\mathcal{U}, G_c)$. We may simply define fiber bundles given by cocycles with respect to different coverings, but represent the same cohomology class to be *isomorphic* to each other. Hence the isomorphism classes of fiber bundles with the base manifold M, fiber F and structure group G are in one-one correspondance to the cohomology classes in $H^1(M, G_c)$. Fiber bundles in the isomorphism class correspond to $\xi \in H^1(X, G_c)$ are said to be *associated to* ξ and elements in $H^1(X, G_c)$ are called *G-bundles*.

Given a vector bundle W over a complex manifold M, with the projection π and the trivialization

$$\phi_U : \pi^{-1}U \longrightarrow U \times \mathbb{C}^r.$$

$F \longrightarrow M$ is a *subbundle* of $E \longrightarrow M$ if each $F_x = \pi^{-1}(x), \forall x \in M$ is a subspace of E_x is with the trivialization

$$\phi_U|_{F_U} : \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^s, \quad s \leq r.$$

and the bunde $E/F \longrightarrow M$ is defined by letting the fibers over $x \in M$, $(E/F)_x = E_x/F_x$ with the induced trivialization

$$\phi_U|_{E/F_U} : \pi^{-1}(U)/F_U \longrightarrow U \times \mathbb{C}^{r-s}$$

be given by $\phi_U|_{E/F_U} : (x, v + F_U) \mapsto (x, v')$ for all $x \in U$, $v \in E_U$ and v' is the subvector of v whose components are from the vector space $\mathbb{C}^{r-s}$.

2. Metrics, Connections and Curvatures of Vector Bundles

We have already defined metrics on differentiable manifolds, namely Riemannian metric, Hermitian metric and Kähler metric in Chapter 1. This section will define Hermitian metrics and then connections and curvatures for vector bundles in general. The main references are [Griffiths], [Mok] and [Kobayashi].

2.1. Hermitian Metrics.

Definition 2.1 (Hermitian Inner Product). A *Hermitian inner product* on a complex vector space V is a complex-valued function $(\cdot, \cdot) : V \times V \longrightarrow \mathbb{C}$, such that

(i) $(a_1\xi_1 + a_2\xi_2, \eta) = a_1(\xi_1, \eta) + a_2(\xi_2, \eta), \quad a_1, a_2 \in \mathbb{C}, \xi_1, \xi_2, \eta \in V.$
(ii) $(\eta, \xi) = \overline{(\xi, \eta)}, \quad \xi, \eta \in V$

(iii) $(\xi, \xi) > 0, \quad \xi \neq 0 \in V$.

Definition 2.2 (Hermitian Metric). Let $E \longrightarrow M$ be a complex vector bundle. A *Hermitian metric* $(\cdot,\cdot)$ on E is defined by assigning a Hermitian inner product $(\cdot,\cdot)_x$ on each fiber E_x of E, varying smoothly with $x \in M$.

Definition 2.3 (Hermitian Vector Bundle). A complex holomorphic vector bundle associated with a Hermitian metric is called a *Hermitian vector bundle*.

Let E be a Hermitian vector bundle of rank r with Hermitian metric $(\cdot,\cdot)$. Suppose $e = \{e_1, \ldots, e_r\}$ is a holomorphic frame of E with corresponding coordinates $(u_1, \ldots, u_r)$. Write $h_{\alpha\beta} = (e_\alpha, \bar{e}_\beta)$, then the corresponding $r \times r$ Hermitian matrix h is defined in the way that the (α, β)-th entry is $h_{\alpha\beta}$ and the *Hermitian form* is defined by

$$H = \sum_{\alpha,\beta} h_{\alpha\beta} du^\alpha \otimes d\bar{u}^\beta.$$

2.2. Connections.

2.2.1. *Connections of Complex Vector Bundles.* Notations: Let $E \longrightarrow M$ be a complex vector bundle over a complex manifold M. Denote as $A^0(E)$ the set of sections of E and $A^p(E)$ for $p \geq 1$ as the set of E-valued differentiable p-forms on M. That is, $A^p(E) = A^p(M, \mathbb{C}) \otimes A^0(E)$.

Definition 2.4 (Connection). A *connection* D on a complex vector bundle $E \longrightarrow M$ is a map

$$D : A^0(E) \longrightarrow A^1(E),$$

satisfying the *Leibniz's rule*, $D(f \cdot \zeta) = df \otimes \zeta + f \cdot D(\zeta)$, for all $\zeta \in A^0(E)$, $f \in C^\infty(M, \mathbb{R})$.

Let $E \longrightarrow M$ be a complex vector bundle of rank r over a complex manifold with Hermitian metric $(\cdot,\cdot)$ and connection D. Suppose $e = \{e_1, \ldots, e_r\}$ be a frame for E. Then since elements in $A^1(M, \mathbb{C}) \otimes A^0(E)$ can be written uniquely as $\sum_\alpha \tau^\alpha \otimes e_\alpha$ for $\tau^\alpha \in A^1(M, \mathbb{C})$, we may write

$$De_\alpha = \sum_\beta {\Gamma_\alpha}^\beta \otimes e_\beta.$$

The matrix $\theta_e = ({\Gamma_\alpha}^\beta)$ with ${\Gamma_\alpha}^\beta$ as the (α, β)-th entry, is the *connection matrix* of D with respect to the frame e. Define the conjugation $\overline{{\Gamma_\alpha}^\beta}$ of ${\Gamma_\alpha}^\beta$ by requiring

$$D(\bar{e}_\alpha) = \sum \overline{{\Gamma_\alpha}^\beta} \otimes \bar{e}_\beta$$

for all $\alpha, \beta \in \{1, \ldots, n\}$. We will introduce the Christoffel symbols. Let $(z_1, \ldots, z_n)$ be coordinates of M. Observe that for any $\zeta \in A^0(E)$, $D(\zeta) \in A^1(E) = A^1(M, \mathbb{C}) \otimes A^0(E)$, and hence $D(\zeta)$ can be viewed as a tensor

$$D(\zeta) : TM \longrightarrow \mathbb{C} \otimes A^0(E) = A^0(E).$$

We denote $D(\zeta)(\eta)$ by $D_\eta(\zeta)$ for any $\eta \in TM$. Under the local frames $\{\frac{\partial}{\partial z_i}\}_{i=1}^n$ of TM and for any $k = 1, \ldots, n$, write

$$D_k(\zeta) = D_{\frac{\partial}{\partial z_k}}(\zeta).$$

In terms of the frame e of E, write

$$D_k(e_\alpha) = \sum_\beta \Gamma_{k\alpha}^\beta \otimes e_\beta$$

where $\Gamma_{k\alpha}^\beta = \Gamma_\alpha{}^\beta\left(\frac{\partial}{\partial z_k}\right), k = 1, \ldots, n$. The $\Gamma_{k\alpha}^\beta$'s are called *Christoffel symbols.* With this notation we can write

$$D(e_\alpha) = \sum_{\beta,i} \Gamma_{i\alpha}^\beta dz^i \otimes e_\beta = \sum_\beta \Gamma_\alpha{}^\beta e_\beta$$

where $\Gamma_\alpha{}^\beta = \sum_i \Gamma_{i\alpha}^\beta dz^i$.

2.2.2. *Hermitian Connections.* We will study connections of a Hermitian vector bundle. There is in general various ways to find connections of a Hermitian vector bundle. We will show that a connection will be uniquely given if it is both compatible with the Hermitian metric and the complex structure. This unique connection will be called the *Hermitian connection* of the Hermitian vector bundle. We will first define two operators ∂ and $\bar{\partial}$. Let M be a complex manifold. We have the decomposition of the tangent space due to the complex structure J, thus we get a decomposition of the cotangent space to M at each point $z \in M$,

$$T_{\mathbb{C},z}^* M = T_z^{1,0*} M \oplus T_z^{0,1*} M$$

in which each T^* is the dual of T. This decomposition gives a further decomposition of the wedge products of the cotangent spaces. Precisely,

$$\wedge^n T_{\mathbb{C},z}^* M = \wedge^n (T_z^{1,0*} M \oplus T_z^{0,1*} M) = \oplus_{p+q=n} (\wedge^p T_z^{1,0*} M \otimes \wedge^q T_z^{0,1*} M).$$

Write

$$\Omega^{p,q}(M) = \{\eta \in \Omega^n(M) : \eta(z) \in \wedge^p T_z^{1,0*} M \otimes \wedge^q T_z^{0,1*} M, z \in M\},$$

and get

$$\Omega^n(M) = \oplus_{p+q=n} \Omega^{p,q}(M).$$

Forms in $\Omega^{p,q}(M)$ are said to be *of type* (p,q). We can also decompose the exterior differential operator d as follows: Let $\Omega^*(M) = \oplus_n \Omega^n(M)$ define the projection

$$\pi^{(p,q)} : \Omega^*(M) \longrightarrow \Omega^{p,q}(M).$$

Denote $\pi^{(p,q)}\eta$ by $\eta^{(p,q)}$. Observe that, for any $\eta \in \Omega^{p,q}(M)$,

$$d\eta(z) \in (\wedge^p T_z^{1,0*} M \otimes \wedge^q T_z^{0,1*} M) \wedge T_{\mathbb{C},z}^* M, \quad z \in M,$$

where the right hand side of "$\in$" equals

$$\begin{aligned} & (\wedge^p T_z^{1,0*} M \otimes \wedge^q T_z^{0,1*} M) \wedge (T_z^{1,0*} M \oplus T_z^{0,1*} M) \\ = \; & (\wedge^p T_z^{1,0*} M \otimes \wedge^q T_z^{0,1*} M) \wedge (T_z^{1,0*} M) \oplus (\wedge^p T_z^{1,0*} M \otimes \wedge^q T_z^{0,1*} M) \wedge (T_z^{0,1*} M) \\ = \; & (\wedge^{p+1} T_z^{1,0*} M \otimes \wedge^q T_z^{0,1*} M) \wedge (T_z^{1,0*} M) \oplus (\wedge^{q+1} T_z^{0,1*} M \otimes \wedge^p T_z^{1,0*} M) \end{aligned}$$

i.e. $d\eta \in \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$.

DEFINITION 2.5 ($\partial, \bar{\partial}$). Define the operators

$$\bar{\partial} : \Omega^{p,q}(M) \longrightarrow \Omega^{p,q+1}(M);$$

$$\partial : \Omega^{p,q}(M) \longrightarrow \Omega^{p+1,q}(M)$$

by $\bar{\partial} = \pi^{(p,q+1)} \circ d$ and $\partial = \pi^{(p+1,q)} \circ d$, respectively. Therefore, $d = \partial + \bar{\partial}$.

For any Hermitian vector bundle $E \longrightarrow M$, we may define $\Omega^0(E)$ the set of holomorphic sections of E. $\Omega^{(p,q)}(E) = \Omega^{(p,q)}(M) \otimes \Omega^0(E)$ and $\Omega^n(E) = \Omega^n(M) \otimes \Omega^0(E)$. There is also induced ∂_E and $\bar{\partial}_E$ defined on $\Omega^{p,q}(E)$ by simply regarding holomorphic sections in E as coefficients.

Let $E \longrightarrow M$ be a Hermitian vector bundle with Hermitian metric $(\cdot, \cdot)$ and connection D. If D satisfies

$$d(\xi, \eta) = (D\xi, \eta) + (\xi, D\eta), \quad \xi, \eta \in A^0(E),$$

then D is compatible with the Hermitian metric and is called a *metric connection.* If we decompose

$$D = D^{1,0} + D^{0,1}$$

by defining

$$D^{1,0} : \Omega^0(E) \xrightarrow{D} \Omega^1(E) \xrightarrow{\pi^{(1,0)}} \Omega^{1,0}(E);$$

$$D^{0,1} : \Omega^0(E) \xrightarrow{D} \Omega^1(E) \xrightarrow{\pi^{(0,1)}} \Omega^{0,1}(E),$$

and $D^{0,1} = \bar{\partial}_E$, then D is compatible with complex structure and is called a *complex connection.*

DEFINITION 2.6 (Hermitian Connection). If a connection D of a complex Hermitian vector bundle is both metric connection and complex connection, then we call it a *Hermitian connection* of the Hermitian vector bundle.

THEOREM 2.1. *If $E \longrightarrow M$ is a Hermitian vector bundle of rank r with the Hermitian metric $(\cdot, \cdot)$, then there exists a unique Hermitian connection D.*

PROOF. Let $e = \{e_1, \ldots, e_n\}$ be a holomorphic frame for E, then the corresponding Hermitian matrix h of the metric is the matrix with the (α, β)-th entry being $h_{\alpha\beta} = (e_\alpha, \bar{e}_\beta)$. Assume the existence of such Hermitian connection D and let θ_e be its connection matrix. Since D is compatible with metric we have,

$$\begin{aligned}(\partial + \bar{\partial})h_{\alpha\beta} &= d(e_\alpha, \bar{e}_\beta) \\ &= (De_\alpha, \bar{e}_\beta) + (e_\alpha, D\bar{e}_\beta) \\ &= (\sum_\gamma \Gamma_\alpha{}^\gamma e_\gamma, \bar{e}_\beta) + (e_\alpha, \sum_\gamma \overline{\Gamma_\beta{}^\gamma} \bar{e}_\gamma) \\ &= \sum_\gamma \Gamma_\alpha{}^\gamma (e_\gamma, \bar{e}_\beta) + \sum_\gamma \overline{\Gamma_\beta{}^\gamma} (e_\alpha, \bar{e}_\gamma) = \sum_\gamma \Gamma_\alpha{}^\gamma h_{\gamma\beta} + \sum_\gamma \overline{\Gamma_\beta{}^\gamma} h_{\alpha\gamma}.\end{aligned}$$

Since D is compatible with the complex structure, we have $\Gamma_\alpha{}^\gamma h_{\gamma\beta}$ is of type $(1,0)$ and $\overline{\Gamma_\beta{}^\gamma} h_{\alpha\gamma}$ is of type $(0,1)$. Comparing types, we have

$$\begin{aligned}\partial h_{\alpha\beta} &= \sum_\gamma \Gamma^\gamma_\alpha h_{\gamma\beta}, \quad \text{i.e. } \partial h = \theta_e h \\ \bar\partial h_{\alpha\beta} &= \sum \overline{\Gamma_\beta{}^\gamma} h_{\alpha\gamma}, \quad \text{i.e. } \bar\partial h = h^t \bar\theta_e\end{aligned}$$

and we see that this is the unique solution to both equations:

$$\theta_e = \partial h \cdot h^{-1}. \tag{10}$$

□

2.2.3. *Hermitian Connection and Riemannian Connection.* Let M be a Hermitian manifold with Hermitian metric h, the Hermitian connection D of its tangent bundle which is in particular a Hermitian vector bundle. When regarding h as a Riemannian metric of M, we denote it by g and let ∇ be the unique Riemannian connection. The following context will show D agrees with ∇ if and only if the manifold M is Kähler. Observe that D is compatible with the metric g. Indeed,

$$dg(\eta,\xi) = dh(\eta,\xi) = h(D\eta,\xi) + h(\eta, D\xi) = g(D\eta,\xi) + g(\eta, D\xi)$$

for any $\eta,\xi \in T^{\mathbb{C}}M$. Hence the statement is reduced to the following theorem by uniqueness of Riemannian connection.

THEOREM 2.2. *The Hermitian connection D is torsion free, if and only if the Kähler form of the Hermitian metric is closed. Hence D=∇ if and only M is Kähler.*

PROOF. Compute the torsion tensor T of M defined by $T(\eta,\xi) = D_\eta(\xi) - D_\xi(\eta)$ for $\eta,\xi \in T^{\mathbb{C}}M$. If η and ξ are of opposite type, we may assume that η is of type $(1,0)$. Hence,

$$T(\eta,\xi) = D_\eta(\xi) - D_\xi(\eta) = \underbrace{D(\xi)}_{(0,1)-form}\underbrace{(\eta)}_{type(1,0)} + \underbrace{D(\eta)}_{(1,0)-form}\underbrace{(\xi)}_{type(0,1)} = 0.$$

In terms of holomorphic coordinates $\{z_i\}$ of M,let T operate on the holomorphic basis of TM and antiholomorphic basis respectively,

$$T\left(\frac{\partial}{\partial z_i},\frac{\partial}{\partial z_j}\right) = D_i\left(\frac{\partial}{\partial z_j}\right) - D_j\left(\frac{\partial}{\partial z_i}\right) = \sum_k (\Gamma^k_{ij} - \Gamma^k_{ji})\frac{\partial}{\partial z_k}.$$

By taking conjugates,

$$T\left(\frac{\partial}{\partial \bar z^i},\frac{\partial}{\partial \bar z^j}\right) = \sum_k (\overline{\Gamma^k_{ij}} - \overline{\Gamma^k_{ji}})\frac{\partial}{\partial \bar z^k}.$$

It follows that D is torsion free if and only if

$$\Gamma^k_{ij} = \Gamma^k_{ji} \quad \text{for all } i,j \text{ and } k.$$

On the other hand, for any point $P \in M$, choosing holomorphic coordinates $\{z_i\}$ such that $g_{i\bar{j}}(P) = \delta_{ij}$, by the formula (10), we have

$$\Gamma^k_{ij} = \frac{\partial}{\partial z_i} g_{j\bar{k}}.$$

Thus, the Hermitian connection D is torsion free if and only if

$$\frac{\partial}{\partial z_i} g_{j\bar{k}}(P) = \frac{\partial}{\partial z_j} g_{i\bar{k}}(P)$$

that is $d\omega(P) = 0$ for any $P \in M$ which means M is Kähler. □

2.2.4. *Second Fundamental Forms.* We will now study connections of Hermitian vector bundles and subbundles. Let $E \xrightarrow{\pi} M$ be a Hermitian vector bundle of rank r, $S \xrightarrow{\pi} M$ a holomorphic subbundle of rank s and $Q = E/S \xrightarrow{\pi} M$ the quotient bundle. Let h_E be the Hermitian metric of E, $h_S = h_E|S$ be the Hermitian metric of S defined by the restriction of h_E on S and $h_Q = h_E|S^\perp$ be the Hermitian metric of Q defined by the restriction of h_E on $S^\perp$. Let D_E, D_S and D_Q be the Hermitian connections of E, S and Q respectively. Observe that the restriction of D_E on S equals D_S. Indeed, the restriction $D_E|S$ is both compatible with the Hermitian metric h_S and the complex structure of S, thus $D_E|_S$ is the Hermitian connection of S and hence equals D_S by uniqueness of the Hermitian connection as shown in Theorem 2.1. Similarly, we also have $D_E|Q = D_Q$. Fix a frame e of E and write θ_E, θ_S and θ_Q as the connection matrices with respect to e of D_E, D_S and D_Q respectively. Then,

$$\theta_E = \begin{pmatrix} \theta_E|S_{s\times s} & B_{s\times(r-s)} \\ A_{(r-s)\times s} & \theta_E|Q_{(r-s)\times(r-s)} \end{pmatrix} = \begin{pmatrix} \theta_S & B_{s\times(r-s)} \\ A_{(r-s)\times s} & \theta_Q \end{pmatrix} \tag{11}$$

Definition 2.7 (Second Fundamental Form). The $(r-s) \times s$ matrix A in (11) is called the *second fundamental form* of E with respect to S.

For any tangent vector $v \in T_P S$, and $u \in T_P E$, $A(v)(u)$ is the projection of the directional derivative of v with respect to u on to the quotient tangent space $T_P Q$. This operator is actually independent of choices of frames.

2.3. Curvatures.

2.3.1. *Curvatures of Vector Bundles.* Given a connection D of a complex vector bundle $E \longrightarrow M$ of rank r, we may generalize it to an operator for any $p \geq 0$,

$$D_p : A^p(E) \longrightarrow A^{p+1}(E)$$

by requiring the Leibniz's rule, $D_p(\psi \wedge \xi) = d\psi \otimes \xi + (-1)^p \psi \wedge D_p \xi$ for all $\psi \in A^p(M, \mathbb{C}), \xi \in A^0(E)$.

Definition 2.8 (Curvature). Define the *curvature* of the connection D by

$$D^2 = D_1 \circ D_0 : A^0(E) \longrightarrow A^2(E).$$

Since $A^2(E) = A^2(M,\mathbb{C}) \otimes A^0(E)$, under a frame $e = \{e_1, \ldots, e_r\}$ of E elements in $A^2(E)$ can be written uniquely as $\sum_\beta \nu^\beta \otimes e_\beta$ where $\nu^\beta \in A^2(M,\mathbb{C})$. Write

$$D^2 e_\alpha = \sum_\beta {\Theta_\alpha}^\beta \otimes e_\beta.$$

We call the $r \times r$ matrix $\Theta_e = ({\Theta_\alpha}^\beta)$ with the (α,β)-th entry ${\Theta_\alpha}^\beta$ the *curvature matrix* of the curvature D^2 with respect to the frame e. We can represent the curvature matrix Θ_e by the connection matrix θ_e with respect to the frame e,

$$\begin{aligned} D^2 e_\alpha &= D \sum_\beta {\Gamma_\alpha}^\beta \otimes e_\beta \\ &= \sum_\beta D({\Gamma_\alpha}^\beta \otimes e_\beta) \\ &= \sum_\beta d{\Gamma_\alpha}^\beta \otimes e_\beta - {\Gamma_\alpha}^\beta \wedge D(e_\beta) \\ &= \sum_\beta (d{\Gamma_\alpha}^\beta - \sum_\gamma {\Gamma_\alpha}^\gamma \wedge \Gamma_\gamma^\beta) \otimes e_\beta. \end{aligned}$$

That is,

$$\Theta_e = d\theta_e - \theta_e \wedge \theta_e. \tag{12}$$

The identity (12) is called the *Cartan structure equation.* Now we will show how this curvature depends on the frame e. Suppose e' is another frame of E such that $e' = Ae, A = (a_{\alpha\beta}) \in GL(r;\mathbb{C})$, then

$$\begin{aligned} D^2 e'_\alpha &= D^2 \sum_\beta a_{\alpha\beta} e_\beta \\ &= \sum_{\beta,\gamma} a_{\alpha\beta} {\Theta_\beta}^\gamma \otimes e_\gamma = \sum_{\beta,\gamma} a_{\alpha\beta} {\Theta_\beta}^\gamma \otimes a^{-1}_{\gamma\beta} e'_\gamma. \end{aligned}$$

Hence the curvature matrix $\Theta_{e'}$ with respect to e' is given by

$$\Theta_{e'} = A \Theta_e A^{-1}.$$

2.3.2. *Curvatures of Hermitian Connections on Hermitian Vector Bundles.* Let E be a Hermitian vector bundle of rank r over a complex manifold M of dimension n and let D be the Hermitian connection of E. Let e be any holomorphic frame of E. Then we have:

THEOREM 2.3. *Let Θ be the curvature of the Hermitian connection D on the Hermitian vector bundle E be the Hermitian connection. Under the frame e, the corresponding curvature matrix Θ_e is a Hermitian matrix of $End(E)$-valued $(1,1)$-forms on M.*

PROOF. Without loss of generality, we may assume that e is further unitary. D is compatible with the complex structure i.e. $D^{0,1} = \bar\partial$. This implies that $D^{0,1} \cdot D^{0,1} = \bar\partial \cdot \bar\partial = 0$ and hence $\Theta^{0,2} = 0$. Compatibility of D with the Hermitian metric implies that the connection matrix θ_e with respect to the unitary holomorphic frame e is skew-Hermitian. Indeed, $0 = d(e_\alpha, e_\beta) = {\Gamma_\alpha}^\beta +$

$\overline{\Gamma_\beta{}^\alpha}$. Then $\Theta_e = d\theta_e - \theta_e \wedge \theta_e$ is also skew-Hermitian. Thus $\Theta_e^{2,0} = -\overline{\Theta^{0,2}} = 0$. Since the type of Θ_e is invariant under the change of frame, we see that Θ_e is a Hermitian matrix of $(1,1)$-forms. □

Further suppose that $(z_1, \ldots, z_n)$ are coordinates of M, hence we may write $\partial = \sum \frac{\partial}{\partial z_i} dz^i$ and $\bar\partial = \sum \frac{\partial}{\partial \bar z_i} d\bar z^i$. $D^2 e_\alpha = \sum_\beta \Theta_\alpha{}^\beta \otimes e_\beta$ gives the $\Theta_\alpha{}^\beta$'s as $End(E)$-valued $(1,1)$-forms on M by Theorem 2.3. We may write

$$\Theta_\alpha{}^\beta = \sum_{i,j} \Theta_\alpha{}^\beta{}_{i\bar j} dz^i \wedge d\bar z^j, \tag{13}$$

with the scalar $\Theta_\alpha{}^\beta{}_{i\bar j} = \Theta_\alpha{}^\beta \left(\dfrac{\partial}{\partial z_i}, \dfrac{\partial}{\partial \bar z_j}\right)$. For any $\eta = \sum_\alpha a^\alpha e_\alpha \in E$ and $\mu = \sum_\alpha b^\alpha e_\alpha \in E$, $u = \sum_i c_i \frac{\partial}{\partial z_i}$ and $v = \sum_j d_j \frac{\partial}{\partial z_i}$, we may write

$$\Theta_\eta{}^\mu{}_{u\bar v} = \sum_{\alpha,\beta,i,j} a^\alpha b^\beta c_i \bar d_j \Theta_\alpha{}^\beta{}_{i\bar j}.$$

We call the Hermitian vector bundle is *positive (non-negative) in the sense of Griffiths* at $x \in M$, if for any non-zero vector $\eta \in E_x$ and non-zero tangent vector $v \in T_x^{1,0}M$, we have

$$\Theta_{\eta\bar\eta v\bar v} > (\geq) 0.$$

We can further represent the curvature matrix by metric matrix in this case. Let h denote the metric matrix under the frame e. That is $h = (h_{\alpha\beta})$ where $h_{\alpha\beta} = (e_\alpha, \bar e_\beta)$ is the (α,β)-th entry of h. Denote the (α,β)-th entry of h^{-1} by $h^{\alpha,\beta}$.

Proposition 2.1. $\Theta = -\partial\bar\partial h \cdot h^{-1} + \partial h \wedge \bar\partial h \cdot h^{-2}$.

Proof. The proof needs

$$dh^{-1} = -h^{-1} \cdot dh \cdot h^{-1},$$

which can be shown by direct checking. By formula (10), the Cartan structure equation can be computed as follows,

$$\begin{aligned}
\Theta &= d\theta - \theta \wedge \theta \\
&= d(\partial h \cdot h^{-1}) - (\partial h \cdot h^{-1}) \wedge (\partial h \cdot h^{-1}) \\
&= \bar\partial\partial h \cdot h^{-1} - \partial h \wedge dh^{-1} - (\partial h \cdot h^{-1}) \wedge (\partial h \cdot h^{-1}) \\
&= \bar\partial\partial h \cdot h^{-1} - \partial h \wedge (-h^{-1} \cdot dh \cdot h^{-1}) - (\partial h \cdot h^{-1}) \wedge (\partial h \cdot h^{-1}) \\
&= -\partial\bar\partial h \cdot h^{-1} + \partial h \wedge \bar\partial h \cdot h^{-2}.
\end{aligned}$$

Or equivalently,

$$\Theta_\alpha{}^\beta = -\sum_\gamma \partial\bar\partial h_{\alpha\gamma} h^{\gamma\beta} + \sum_{\gamma,\eta,\mu} \partial h_{\alpha\gamma} \bar\partial h_{\gamma\eta} h^{\eta\mu} h^{\mu\beta}. \tag{14}$$

Hence,

$$\Theta_\alpha{}^\beta{}_{i\bar j} = \Theta_\alpha{}^\beta \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar z_j}\right) = -\sum_\gamma \frac{\partial^2 h_{\alpha\gamma}}{\partial z_i \partial \bar z_j} h^{\gamma\beta} + \sum_{\gamma,\eta,\mu} \frac{\partial h_{\alpha\gamma}}{\partial z_i} \frac{\partial h_{\gamma\eta}}{\partial \bar z_j} h^{\eta\mu} h^{\mu\beta}. \tag{15}$$

□

2.3.3. *Curvatures of Tangent Bundles.* Let M be a Hermitian manifold with tangent bundle TM which is in particular a Hermitian vector bundle. Let D be the Hermitian connection and D^2 the corresponding curvature of D. Hence under the local holomorphic coordinates $(z_1, \ldots, z_n)$ of M and the holomorphic frame e of TM defined by $\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\}$, the curvature matrix $\Theta_e = ({\Theta_i}^j)$ of the curvature D^2 can be written as

$$ {\Theta_i}^j = \sum_{k,l} {\Theta_i}^j{}_{k\bar{l}} dz^k \wedge d\bar{z}^l $$

where ${\Theta_i}^j{}_{k\bar{l}} = {\Theta_i}^j \left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l} \right)$ as in (13). Represent the curvature by metric as (14) and (15), we have:

$$ {\Theta_i}^j = -\sum_k \partial\bar{\partial} h_{ik} h^{kj} + \sum_{k,l,m} \partial h_{ik} \bar{\partial} h_{kl} h^{lm} h^{mj}, \tag{16} $$

$$ {\Theta_i}^j{}_{k\bar{l}} = -\sum_m \frac{\partial^2 h_{im}}{\partial z_k \partial \bar{z}_l} h^{mj} + \sum_{m,n,p} \frac{\partial h_{im}}{\partial z_k} \frac{\partial h_{mn}}{\partial \bar{z}_l} h^{np} h^{pj}. \tag{17} $$

Definition 2.9 (Curvature Tensor). Let Θ be the curvature of the Hermitian connection D of a Hermitian manifold M with Hermitian metric h. The *curvature tensor* R of M is a 4-tensor $R : T^{1,0}M \otimes \overline{T^{1,0}M} \otimes T^{1,0}M \otimes \overline{T^{1,0}M} \longrightarrow \mathbb{C}$ defined by

$$ R\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l} \right) = h\left(\Theta \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j} \right) \left(\frac{\partial}{\partial z_k} \right), \frac{\partial}{\partial z_l} \right). $$

Denote $R\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l} \right)$ by $R_{i\bar{j}k\bar{l}}$, then

$$ R = \sum R_{i\bar{j}k\bar{l}} dz^i \otimes d\bar{z}^j \otimes dz^k \otimes d\bar{z}^l $$

There is obvious linear relation between the curvature tensor with the complexified Riemannian curvature tensor under the coordinate changes, say $z_i = x_i + \sqrt{-1} y_i$ and $y_i = x_{i+n}$ for $i = 1, \ldots, n$. Hence the symmetries of curvature tensor of a Riemannian manifold, as stated in Proposition 2.1, remains true when the curvature tensor is of a Hermitian manifold.

If the Hermitian manifold is Kähler. Besides the symmetries of curvature tensors of M as a Riemannian manifold as stated in Proposition 2.1, we have additional properties of curvature tensor when the M is Kähler as stated in [Mok] p. 31:

Proposition 2.2. *If M is a Kähler manifold, then*

$$ R_{i\bar{j}k\bar{l}} = R_{i\bar{l}k\bar{j}}. $$

Represent the curvature tensor by metrics using (17) and note that

$$\Theta = \sum_{r,s} \Theta_r{}^s dz^r \otimes \frac{\partial}{\partial z_s} = \sum_{r,s,p,q} \Theta_r{}^s{}_{p\bar{q}} dz^p \wedge d\bar{z}^q \wedge dz^r \otimes \frac{\partial}{\partial z_s}.$$

We have

$$\begin{aligned} R_{i\bar{j}k\bar{l}} &= h\left(\Theta\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right)\left(\frac{\partial}{\partial z_k}\right), \frac{\partial}{\partial z_l}\right) \\ &= h\left(\sum_{r,s,p,q} \Theta_r{}^s{}_{p\bar{q}} dz^p \wedge d\bar{z}^q \wedge dz^r \otimes \frac{\partial}{\partial z_s}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right)\left(\frac{\partial}{\partial z_k}\right), \frac{\partial}{\partial z_l}\right) \\ &= h\left(\sum_{r,s} \Theta_r{}^s{}_{i\bar{j}} dz^r \otimes \frac{\partial}{\partial z_s}\left(\frac{\partial}{\partial z_k}\right), \frac{\partial}{\partial z_l}\right) \\ &= h\left(\sum_{s} \Theta_k{}^s{}_{i\bar{j}} \frac{\partial}{\partial z_s}, \frac{\partial}{\partial z_l}\right) \\ &= \sum_s \Theta_k{}^s{}_{i\bar{j}} h_{sl} \\ &= \sum_s \left(-\sum_m \frac{\partial^2 h_{km}}{\partial z_i \partial \bar{z}_j} h^{ms} + \sum_{m,n,p} \frac{\partial h_{km}}{\partial z_i}\frac{\partial h_{mn}}{\partial \bar{z}_j} h^{np} h^{ps}\right) h_{sl} \\ &= -\frac{\partial^2 h_{kl}}{\partial z_i \partial \bar{z}_j} + \sum_{m,n} \frac{\partial h_{km}}{\partial z_i}\frac{\partial h_{mn}}{\partial \bar{z}_j} h^{nl}. \end{aligned}$$

That is,

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 h_{kl}}{\partial z_i \partial \bar{z}_j} + \sum_{m,n} \frac{\partial h_{km}}{\partial z_i}\frac{\partial h_{mn}}{\partial \bar{z}_j} h^{nl}. \tag{18}$$

Compare (17) and (18), we have:

$$R_{i\bar{j}k\bar{l}} = \sum_p h_{pj} \Theta_i{}^p{}_{k\bar{l}}. \tag{19}$$

Definition 2.10 (Ricci Curvature). The *Ricci curvature tensor* $Ric_{k\bar{l}} : T_p^{1,0}M \otimes \overline{T_p^{1,0}M} \longrightarrow \mathbb{C}$ is defined to be the contraction of the curvature tensor

$$Ric_{k\bar{l}} = \sum_{i,j} h^{ij} R_{i\bar{j}k\bar{l}} dz^k \otimes d\bar{z}^l.$$

The *Ricci form* is defined by

$$\sum_{k,l} \frac{\sqrt{-1}}{2\pi} Ric_{k\bar{l}} dz^k \wedge d\bar{z}^l.$$

By substituting (18) into $Ric_{k\bar{l}}$, we get:

$$Ric_{k\bar{l}} = \sum_{i,j} h^{ij}\left(-\frac{\partial^2 h_{kl}}{\partial z_i \partial \bar{z}_j} + \sum_{m,n} \frac{\partial h_{km}}{\partial z_i}\frac{\partial h_{mn}}{\partial \bar{z}_j} h^{nl}\right) dz^k \otimes d\bar{z}^l. \tag{20}$$

Note that by (19), we have the coefficient of the Ricci tensor is represented by elements of curvature matrix:

$$Ric_{k\bar{l}} = \sum_{i,j} h^{ij} R_{i\bar{j}k\bar{l}} = \sum_{i,j} h^{ij} \sum_{p} h_{pj} \Theta_i{}^p{}_{k\bar{l}} = \sum_{i} \Theta_i{}^i{}_{k\bar{l}}.$$

Hence the Ricci form is given by

$$\frac{\sqrt{-1}}{2\pi} \sum_{k,l} Ric_{k\bar{l}} dz^k \wedge d\bar{z}^l = \frac{\sqrt{-1}}{2\pi} \sum_{i} \sum_{k,l} \Theta_i{}^i{}_{k\bar{l}} dz^k \wedge d\bar{z}^l = \frac{\sqrt{-1}}{2\pi} Tr(\Theta). \tag{21}$$

Definition 2.11 (Scalar Curvature). The *scalar curvature* K is defined to be the contraction of Ricci curvature tensor

$$K = \sum_{k,l} h^{kl} Ric_{k\bar{l}} = \sum_{k,l} h^{kl} \sum_{i,j} h^{ij} R_{i\bar{j}k\bar{l}}.$$

By substituting (20) into K, we get:

$$K = \sum_{k,l} h^{kl} \sum_{i,j} h^{ij} \left(-\frac{\partial^2 h_{kl}}{\partial z_i \partial \bar{z}_j} + \sum_{m,n} \frac{\partial h_{km}}{\partial z_i} \frac{\partial h_{mn}}{\partial \bar{z}_j} h^{nl} \right) dz^k \otimes d\bar{z}^l. \tag{22}$$

Under the condition of M being Kähler, it is meaningful to define the notion of sectional curvature. Recall that if the complex manifold M is *Kähler* then the Hermitian connection agrees with the Riemannian connection. Notions of sectional curvature in Riemannian geometry can be generalized in this case. We will fix the assumptions in the rest of this section: Let M be a Kähler manifold with Kähler metric $g = (g_{i\bar{j}})$. Let D be the Hermitian connection and Θ be the curvature of D and R be the corresponding curvature tensor in the rest of this section.

Definition 2.12 (Holomorphic Sectional Curvature). Let ξ be of unit length in $T^{1,0}M$, then the *holomorphic sectional curvature in the direction of* ξ is defined by

$$R(\xi, \bar{\xi}, \xi, \bar{\xi}).$$

In coordinates, if $\xi = \sum_i u^i \frac{\partial}{\partial z_i}$, then $R(\xi, \bar{\xi}, \xi, \bar{\xi}) = \sum_{i,j,k,l} R_{i\bar{j}k\bar{l}} u^i \bar{u}^j u^k \bar{u}^l$. We may denote $R(\xi, \bar{\xi}, \xi, \bar{\xi}) = R_{\xi\bar{\xi}\xi\bar{\xi}}$.

Let J denote the complex structure of the Kähler manifold (M, g), for any $u \in T^{\mathbb{R}}M$ define

$$\frac{R(u, Ju, u, Ju)}{g(u, Ju)^2}$$

to be the *complexified Riemannian sectional curvature* of the J-invariant real 2-plane generated by u. The proof of independence of choices of the generator u of the holomorphic sectional curvature of the 2-plane can be found in [Kobayashi] Chapter IV. Its positivity(negativity) is defined similarly as that of Riemannian sectional curvature as in (2). The relation between the holomorphic sectional curvature and the complexified Riemannian sectional curvature is as follows: Recall that we have the natural $\mathbb{R}$-isomorphism between $T^{\mathbb{R}}M \longrightarrow T^{1,0}M$. The inverse is given by $h : \xi \mapsto \xi + \bar{\xi} = 2Re\xi$. For any $\xi \in T^{1,0}M$ of unit length, let $u = h(\xi)$. Then it can be computed that (refer to [Mok] p. 31),

$$\frac{R(u, Ju, u, Ju)}{g(u, Ju)^2} = R_{\xi\bar{\xi}\xi\bar{\xi}}.$$

The holomorphic sectional curvature is *positive(negative)* if as a complexified Riemannian sectional curvature, it is positive(negative). If the holomorphic sectional curvature is constant for all $\xi \in T_P^{1,0}M$ and for all $P \in M$, then M is said to be of *constant holomorphic sectional curvature*. We simply list the following two propositions from [Kobayashi] Chapter IX for later use:

PROPOSITION 2.3. *Let M be a connected Kähler manifold of complex dimension $n \geq 2$. If the holomorphic sectional curvature $R_{\xi\bar{\xi}\xi\bar{\xi}}$ depends only on $P \in M$ for any unit $\xi \in T_P^{1,0}M$, then M is of constant holomorphic sectional curvature.*

PROPOSITION 2.4. *A Kähler manifold M of Kähler metric g is of constant holomorphic sectional curvature if and only if*

$$R_{i\bar{j}k\bar{l}} = C(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{\bar{j}k}).$$

Proposition 2.3 and 2.4 imply:

COROLLARY 2.5. *Let M be a connected Kähler manifold of complex dimension $n \geq 2$. If the holomorphic sectional curvature $R_{\xi\bar{\xi}\xi\bar{\xi}}$ for any unit $\xi \in T_P^{1,0}M$ depends only on $P \in M$, then*

$$R_{i\bar{j}k\bar{l}} = C(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{\bar{j}k}).$$

DEFINITION 2.13 (Holomorphic Bisectional Curvature). Let (M, g) be a Kähler manifold, $P \in M$ an arbitrary point and $\xi, \eta \in T_x^{1,0}M$ are of unit length, then the *holomorphic bisectional curvature* in the direction (ξ, η) is defined by

$$R(\xi, \bar{\xi}, \eta, \bar{\eta}).$$

We denote $R(\xi, \bar{\xi}, \eta, \bar{\eta}) = R_{\xi\bar{\xi}\eta\bar{\eta}}$.

The relation between the holomorphic bisectional curvature and the complexified Riemannian curvature tensor is as follows: Recall that we have the natural $\mathbb{R}$-isomorphism between $T^{\mathbb{R}}M \longrightarrow T^{1,0}M$. The inverse is given by $h : \xi \mapsto \xi + \bar{\xi} = 2Re\xi$. It can be computed that for any unit $\xi, \eta \in T^{1,0}M$ if we write $u = 2Re\xi$ and $v = 2Re\eta$, then

$$\frac{R(u, Ju, v, Jv)}{|u|^2|v|^2} = R_{\xi\bar{\xi}\eta\bar{\eta}}$$

where $|\cdot|$ is the norm induced by the metric.

REMARK 2.6. Compare with holomorphic sectional curvature of a Kähler manifold and the Riemannian sectional curvature of the same Kähler manifold as a Riemannian manifold. The notion of holomorphic sectional curvature is weaker in the sense that it only concerns the Gaussian curvature of surfaces which is the images of planes generated by a vector and its conjugate pair under the exponential map instead of in general two arbitrary vectors. The notion of holomorphic bisectional curvature if seen as a generalization of the holomorphic sectional curvature, two arbitrary vectors u and v do involved. However, in a complex manifold, the interesting curvature may not an arbitrary one but some carries the contribution of the complex structure such as the bisectional curvature concerns u, v, Ju and Jv. The fact is that the resulting biholomorphic sectional curvature is determined by the sum of Riemannian sectional curvature of u, v and that of u, Jv, or equivalently, the sum of Riemannian sectional curvature of u, v and Ju, v.

A holomorphic bisectional curvature is said to be *positive(negative) in the sense of Griffiths* if for all unit $\xi, \eta \in T^{1,0}M$, $R_{\xi\bar{\xi}\eta\bar{\eta}} > (<) 0$.

2.4. Example: Curvature of Fubini-Study Metric. Recall that on the projective space $\mathbb{P}^n_{\mathbb{C}}$, there is the Fubini-Study metric as given in (3):

$$g_{FS} = \sum g_{i\bar{j}} dz^i \otimes d\bar{z}^j$$

where

$$g_{i\bar{j}} = \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log(1 + |z_1|^2 + \cdots + |z_n|^2).$$

By identity (18), the corresponding curvature tensor $R = \sum R_{i\bar{j}k\bar{l}} dz^i \otimes d\bar{z}^j \otimes dz^k d\bar{z}^l$ of g_{FS} is given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + \sum_{m,n} \frac{\partial g_{k\bar{m}}}{\partial z_i} \frac{\partial g_{m\bar{n}}}{\partial \bar{z}_j} g^{n\bar{l}} \tag{23}$$

By identity (20), the corresponding Ricci curvature tensor Ric_{kl} is given by

$$Ric_{k\bar{l}} = \sum_{i,j} g^{i\bar{j}} \left(-\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + \sum_{m,n} \frac{\partial g_{k\bar{m}}}{\partial z_i} \frac{\partial g_{m\bar{n}}}{\partial \bar{z}_j} g^{n\bar{l}} \right) dz^k \otimes d\bar{z}^l. \tag{24}$$

By identity (22), the corresponding scalar curvature K is given by

$$K = \sum_{k,l} g^{k\bar{l}} \sum_{i,j} g^{i\bar{j}} \left(-\frac{\partial^2 g_{k\bar{l}}}{\partial z_i \partial \bar{z}_j} + \sum_{m,n} \frac{\partial g_{k\bar{m}}}{\partial z_i} \frac{\partial g_{m\bar{n}}}{\partial \bar{z}_j} g^{n\bar{l}} \right) dz^k \otimes d\bar{z}^l. \tag{25}$$

3. Characteristic Classes of Vector Bundles

This section is devoted to characteristic classes of vector bundles. We will first give a historical review on development of the characteristic classes in which the main references are [Chern1], [Chern2], [Bor-Hir], [Milnor], [Hir] and [Griffiths]. Secondly, we will give the method of decomposing vector bundles from [Hir], which is useful in later context. Thirdly, we will give the definitions of the Chern classes in the Čech cohomology and the de Rham cohomology.

3.1. Historical Review. We will sketch the process of getting the definitions of characteristic classes of vector bundles in this subsection. In most cases, we will only give statements without proofs. The main references are [Chern1], [Chern2], [Milnor], [Hir] and [Griffiths].

3.1.1. *Stiefel-Whitney Classes of Differentiable Manifolds.* This part is mainly based on [Chern2] Appendix. As shown in Chapter 1, the Poincaré-Hopf Theorem, saying that

$$\chi(M) = Index(X)$$

for a compact oriented differentiable manifold M of dimension n and any smooth vector field with only isolated singularities X over M, provides a differential geometrical way to calculate the Euler characteristic. This method is generalized by Whitney. Instead of applying only one vector field, we may apply k vector fields. Suppose $X_1, \ldots, X_k, k \leq n$ are k smooth vector fields on the compact manifold M such that at most points on M the k vector fields are linearly independent and at points where they are linearly dependent, the vector space spanned by them is of dimension $(k-1)$. Since the vector fields are all smooth, $S(X_1, \ldots, X_k)$ defined to be the set of all points in M where the k vector fields are linearly dependent, will be a submanifold of dimension $(k-1)$ of M. Use the language of singular homology, $S(X_1, \ldots, X_k)$ is a singular $(k-1)$-chain in $C_{k-1}(M, \mathbb{Z})$. Furthermore, $S(X_1, \ldots, X_k)$ can be shown to be closed and hence is in $Z_{k-1}(M, \mathbb{Z})$. $S(X_1, \ldots, X_k)$ can thus represent a $(k-1)$-singular homology class $[S(X_1, \ldots, X_k)] \in H_{k-1}(M, \mathbb{Z})$. The fact is that if $Y_1, \ldots, Y_k$ are another choices of such vector fields and gives a $(k-1)$ singular homology class $[S(Y_1, \ldots, Y_k)] \in H_{k-1}(M, \mathbb{Z})$ then $[S(Y_1, \ldots, Y_k)] = [S(X_1, \ldots, X_k)]$. That is $S_X - S_Y = \partial T$ for some $T \in C_k(M, \mathbb{Z})$ where S_X denotes $S(X_1, \ldots, X_k)$ and S_Y denotes $S(Y_1, \ldots, Y_k)$. In generic case, T can be obtained as follows: Define T_1 to be the set of integration curves of vector fields spanned by $\{Y_1, \ldots, Y_k\}$ starting from points in S_X and define T_2 to be the set of integration curves of vector fields spanned by $\{X_1, \ldots, X_k\}$ starting from points in S_Y. Then T can be defined as the intersection $T_1 \cap T_2$. Hence we may denote $S(X_1, \ldots, X_k)$ by S_{k-1}. Let $[w^{n-k+1}]$ denote the Poincaré dual of $[S_{k-1}]$ in the cohomology class $H^{n-k+1}(M, \mathbb{Z})$, this is the $(n-k+1)$*-th Stiefel-Whitney classes* of M. It turns out that when $k = 1$,

$$\chi(M) = \int_M [w^n], \tag{26}$$

where $\int_M$ is the pairing with the *fundamental homology class* $[M]$, which is determined by the orientation of M. Hence the Euler characteristic of a compact oriented differentiable manifold M can be represented by the pairing between the n-th Stiefel-Whitney class and the fundamental homology class $[M]$ of M. This is a hint for getting topological invariants from calculating some cohomology classes.

3.1.2. *Characteristic Classes of Vector Bundles.* The notion of Whitney classes is generalized to be the cohomology class of a vector bundle instead of only tangent bundle by Whitney. Consider firstly a basic fiber bundle as follows:

$$\pi : S(r,q;\mathbb{R}) \longrightarrow G(r,q;\mathbb{R}), \quad 0 \leq r \leq q$$

where the $S(r,q;\mathbb{R})$ is the set of all r-frames which is locally linearly independent r-tuples in the Euclidean space $\mathbb{R}^q$ and $G(r,q;\mathbb{R})$ is the *real Grassmannian manifold*, which is the set of all r-planes in $\mathbb{R}^q$ through the origin. There is a proof in [Milnor] of the Grassmannian manifold being a differentiable manifold. The projection π maps each r-frame to the r-plane through origin spanned by it. To describe points in a Grassmannian manifold, we induce Schubert symbol from [Milnor]. Choose a *flag* of $\mathbb{R}^q$ given by a sequence of vector spaces:

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_q = \mathbb{R}^q.$$

Consider the intersection of the flag with any r-plane, say Λ, in the Grassmannian manifold:

$$\{0\} = V_0 \cap \Lambda \subset V_1 \cap \Lambda \subset \cdots \subset V_q \cap \Lambda = \Lambda.$$

Observe that: (i) Each stage is a linear space in Λ with certain dimension. (ii) The jump of dimension is at most 1 in each stage. (iii) The total number of jumps must be equal to r. Therefore any r-plane Λ gives a unique increasing sequence of integers ranging from 1 to q:

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_r),$$

where σ_i means the σ_i-th stage where the i-th jump occurs for $i = 1, \ldots, r$. The σ is called the *Schubert symbol*. Define $e(\sigma)$ be the set of all r-planes in the Grassmannian manifold whose Schubert symbols are exactly σ. Define $\overline{e(\sigma)}$ to be the set of all r-planes whose Schubert symbols $\sigma' = (\sigma'_1, \ldots, \sigma'_r)$ satisfying $\sigma'_i \leq \sigma_i$ for $i = 1, \ldots, r$. That is, the set of planes whose jumps occurs earlier stage-wisely than those with the Schubert symbol σ. The set $\overline{e(\sigma)}$ is proved to be closed and hence represent a homology class $[\overline{e(\sigma)}]$ in the homology class of the Grassmannian with coefficient in $\mathbb{Z}$. Further the homology class can be proved to be independent of the flag chosen.

For any fixed $i = 1, \ldots, r$, define

$$\sigma_{(i)} = \{\sigma = (\sigma_1, \ldots, \sigma_i, \ldots, \sigma_r) : \sigma_i \leq q - r + i - 1\}.$$

Hence $[\overline{e(\sigma_{(i)})}]$ will be a homology class in $H_{r(q-r)-i}(G(r,q;\mathbb{R}), \mathbb{Z})$ which is independent of the choices of flags. The Poincaré dual of the homology class $[\overline{e(\sigma_{(i)})}]$ denoted as $w^i \in H^i(G(r,q;\mathbb{R}), \mathbb{Z})$ is called the *i-th universal Stiefel-Whitney class of* $G(r,q;\mathbb{R})$. Due to the following embedding theorem, which we will come to later, w^i will induce a characteristic class on a vector bundle.

Theorem 3.1 (Whitney-Pontrjagin Imbedding Theorem). *Let M be a finite cell complex. A vector bundle E of rank r over M can be induced by a continuous mapping $f : M \longrightarrow G(r,q)$, $dimM < q - r$, and the f is defined up to a homotopy. Here $G(r,q)$ stands for either the real Grassmannian manifold or the complex one.*

For any fixed $i = 1, \ldots, r$, we call any element $u \in H^i(G(r,q;\mathbb{R}), \mathbb{Z})$ an *i-th universal class of* $G(r,q;\mathbb{R})$. Since E is determined by f in Theorem 3.1, then the pull back $f^*u \in H^i(M, \mathbb{Z})$ is a class only depend on the bundle E. It is called the *i-th characteristic class of* E corresponding to the universal class u. In particular, the pull back f^*w^i of the i-th universal Stiefel-Whitney class w^i is called the *i-th Stiefel-Whitney class of* E. Analogue to the real Grassmannian manifold, S. S. Chern is able to use complex Grassmannian manifold to give characteristic classes of complex vector bundles, that is, the *Chern classes* $c_i(E) \in H^{2i}(M, \mathbb{Z})$ in [Chern1]. It is true that when the complex vector bundle is the tangent bundle of a compact complex manifold M of complex dimension n,

$$\chi(M) = \int_M c_n(M),$$

where $c_n(M) = c_n(TM)$. This is an analogue to the identity (26). With essentially the same idea, that is, by pulling back characteristic classes of the universal bundle to the vector bundle we considered, the Chern classes can be defined equivalently in the language of Čech cohomology. Before that, we will devote the next subsection to some preliminary on decomposition of fiber bundles from [Hir].

3.2. Methods of Decomposition of Fiber Bundles. This is a technique used in [Hir] to define the Chern classes and also to prove the Riemann-Roch theorem.

3.2.1. *Algebraically Constructions.* Let M be a complex manifold. $E \longrightarrow M$ a fiber bundle with structure group G a complex Lie group associated with the G-bundle ξ. Let P be the principal bundle with structure group G and fiber G over M associated to the G-bundle ξ. If $\mathcal{U} = \{U_i\}_{i \in I}$ is an open covering of M and let

$$h_i : \pi^{-1}(U_i) \longrightarrow U_i \times G, \quad i \in I$$

be the trivialization and the corresponding transition functions, representing ξ by

$$g_{ij} : U_i \cap U_j \longrightarrow G.$$

Let G' be a complex Lie subgroup of G, then the induced fiber bundle $P/G' \xrightarrow{\rho} M$ of $P \xrightarrow{\pi} M$ is defined such that each fiber P/G'_x over $x \in M$ is the identification space of the fiber P_x over x with the equivalence relation $(x, v_x) \sim (x, v_x a'), v_x \in P_x a' \in G'$. We denote the equivalence classes by $(x, v_x G')$. The fiber bundle $P/G' \longrightarrow M$ is associated to the same G-bundle ξ. Indeed, fix the same open covering $\mathcal{U}$ for M, the trivialization $H_i : \rho^{-1}(U_i) \longrightarrow U_i \times G/G'$ is induced by $H_i(x, v_x G') = (x, h_i(x, v_x)G')$. The corresponding transition functions give

$$g'_{ij}(x)(v_x G') = H_j H_i^{-1}(x, v_x G') = (x, h_j h_i{-}1(v_x)G') = g_{ij}(x)(v_x)G'.$$

Let $\sigma : P \xrightarrow{\sigma} P/G'$ be the natural bundle map defined by $\sigma(x, v_x) = (x, v_x G')$.

$$\begin{array}{ccc} P & \xrightarrow{\sigma} & P/G' \\ \pi\downarrow & & \downarrow\rho \\ M & \longrightarrow & M \end{array}$$

Note that $\sigma : P \longrightarrow P/G'$ can be regarded as a principal bundle with fiber G' and structure group G'. In fact, we choose the trivial open covering $\mathcal{W} = \{P/G'\}$ for the base space P/G', and define the trivialization

$$t : \sigma^{-1}(P/G') \longrightarrow P/G' \times G'$$

by $t((x, v_xG'), v_xa') = (x, v_xG', a')$ where $(x, v_xG') \in P/G'$ and $v_xa' \in$ the fiber $P/G'_{(x,v_xG')}$. The transition function here is simply the identity element, so the bundle can be viewed as a G'-bundle or a G-bundle. Denote $\tilde{\xi}$ the associated G'-bundle of $P \xrightarrow{\sigma} P/G'$.

On the one hand we can induce from $P \xrightarrow{\sigma} P/G'$ a principal bundle $W \longrightarrow P/G'$ with structure group and fiber G by a bundle map called h as follows. Observe that there is a natural isomorphism $P \cong (P \times G')/G'$ between bundles over P/G' given by

$$([x], v_xa') \mapsto ([x], [x \times a']_{G'})$$

where $[\ \ ]_{G'}$ denotes the equivalence class under the equivalence relation $(x, a') \sim (xb'^{-1}, b'a')$ for any $a', b' \in G'$. Observe also that there is a natural isomorphism $P \cong (P/G') \times G'$ between bundles over P/G' given by

$$([x], v_xa') \mapsto ([x], v_xG' \times a').$$

Similarly, by replacing all $a' \in G'$ by $a \in G$, define

$$W \xrightarrow{\sigma'} P/G'$$

by $W = P \times G/G'$, since it is isomorphic to $P/G' \times G$. The associated G'-bundle $\tilde{\xi}$ of $P \longrightarrow P/G'$ can be induced canonically to the associated G-bundle $h\tilde{\xi}$ of $W \longrightarrow P/G'$. Therefore W is a principal bundle with structure group and fiber G associated to the G-bundle $h\tilde{\xi}$.

On the other hand, let P denote the underlying topological space of the fiber bundle $P \xrightarrow{\pi} M$ and P/G' denote the underlying topological space of the fiber bundle $P/G' \xrightarrow{\rho} M$. Now consider the induced bundle

$$\rho^*(P) \longrightarrow P/G'$$

from the fiber bundle $P \longrightarrow M$ by the mapping $\rho : P/G' \longrightarrow M$.

$$\begin{array}{ccc} P & \xrightarrow{\rho^*} & \rho^*(P) \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \rho^*\pi} \\ M & \xleftarrow{\rho} & P/G' \end{array}$$

That is, $\rho^*(P) \longrightarrow P/G'$ is a subbundle of the trivial bundle $P/G' \times P \longrightarrow P/G'$ defined by

$$\rho^*(P) = \{((x, v_xG'), (x, u_x)) \in P/G' \times P | v_x, u_x \in P_x\}.$$

This is a fiber bundle with fiber G and the transition functions $\{\tilde{g}_{ij}\}$ representing $\rho^*\xi$ of $\rho^*(P) \longrightarrow P/G'$ are identities in the group G. The trivializations $\tilde{H}_i : (\rho^*\pi)^{-1}(V_i) \longrightarrow V_i \times G$ with respect to the open covering $\{V_i = \rho^{-1}(U_i)\}$ is given by $\tilde{H}_i((x, v_xG'), (x, u_x)) = ((x, v_xG'), h_i(u_x))$. Hence $\tilde{g}_{ij}(x, v_xG')(u_x) = h_jh_i^{-1}(x, v_xG')(u_x) = id(u_x)$.

Since $\rho^*(P) \longrightarrow P/G'$ and $W \longrightarrow P/G'$ are both principal bundle with fiber G and identity transition functions, they are isomorphic fiber bundles. In summary, we have:

THEOREM 3.2.

$$h\tilde{\xi} = \rho^*\xi.$$

This theorem says that $\rho^*\xi$ as a G-bundle over P/G' can always be reduced to the G'-bundle $\tilde{\xi}$ over P/G'. The following theorem gives the condition when ξ a G-bundle over M can be reduced to a G'-bundle over M. We will state without proof the theorem from [Hir] p. 45.

THEOREM 3.3. *A G-bundle ξ over M can be reduced to a G'-bundle η over M, i.e. $h(\eta) = \xi$ if and only if the fiber bundle P/G' has a section s. If this is the case, then $\eta = s^*\tilde{\xi}$.*

3.2.2. *The Building Blocks of the Decompositions.* We will mainly give two kinds of building blocks: (i) The complex Grassmannian manifolds. (ii) The manifolds of flags.

(i) The complex Grassmannian manifold $G(r, q; \mathbb{C})$, which we will denote as $G(r, q)$ in the following context, is the manifold of all r-planes through the origin in $\mathbb{C}^q$. $G(r, q)$ can be represented in terms of matrices: The group $GL(q, \mathbb{C})$ acts transitively on $G(r, q)$. Pick a point of $G(r, q)$, say L_r, determined by $z_{r+1} = z_{r+2} = \cdots = z_q = 0$ where $\{z_1, \dots, z_q\}$ are the local coordinates of $\mathbb{C}^q$. The stabilizer of L_r is the subgroup $GL(r, q-r; \mathbb{C})$ of $GL(q, \mathbb{C})$ consisting all $q \times q$-matrices which maps the space L_r to itself, any $A \in GL(r, q-r; \mathbb{C})$ can be written in the form

$$A = \begin{pmatrix} A' & B \\ 0 & A'' \end{pmatrix}$$

where $A' \in GL(r, \mathbb{C})$, $A'' \in GL(q-r, \mathbb{C})$ and B is any $r \times (q-r)$-matrices with complex number as elements. Therefore we have

$$G(r, q) = GL(q, \mathbb{C})/GL(r, q-r; \mathbb{C}).$$

(ii) A *flag* in the vector space $\mathbb{C}^q$ is a sequence of complex vector spaces in $\mathbb{C}^q$:

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{q-1} \subset V_q = \mathbb{C}^q,$$

where $V_i, i = 1, \dots, q$ is of complex dimension i. We denote $F(q)$ to be the *manifold of flags* in $\mathbb{C}^q$. The flag manifold $F(q)$ can also be represented in terms of matrices: the group $GL(q, \mathbb{C})$ acts transitively on $F(q)$. Pick a flag in $F(q)$, say

$$L = (\{0\} = L_0 \subset L_1 \subset \cdots \subset L_q = \mathbb{C}^q)$$

where L_r is determined by $z_{r+1} = z_{r+2} = \cdots = z_q = 0$ for $r = 1, \ldots, q$. The stablizer group $\Delta(q, \mathbb{C})$ of the flag L is therefore consists of $q \times q$-matrices which preserve every L_r for $r = 1, \ldots, q$. That is

$$\Delta(q, \mathbb{C}) = \cap_{r=1}^{q} GL(r, q-r; \mathbb{C}).$$

Note that matrices in $\Delta(q, \mathbb{C})$ are all diagonal matrices. The manifold of flags can be written as

$$F(q) = GL(q, \mathbb{C})/\Delta(q, \mathbb{C}).$$

3.2.3. *Decompositions of Principal $GL(r, q-r; \mathbb{C})$-bundles and Principal $\Delta(q, \mathbb{C})$-bundles.* (i) Let M be a complex manifold and $G = GL(q, \mathbb{C})$. Suppose that $W \longrightarrow M$ is a fiber bundle with the structure group G and is associated to the G-bundle ξ. Let $P \longrightarrow M$ be the principal bundle with structure group and fiber G and be associated to ξ. Let $G' = GL(r, q-r; \mathbb{C})$ be a subgroup of G. By the constructions in the proof of Theorem 3.2, we have the principal G'-bundle $P \xrightarrow{\tilde{\xi}} P/G'$, which is associated to $\tilde{\xi}$. The natural homomorphism $G' \longrightarrow GL(r, \mathbb{C}) \times GL(q-r, \mathbb{C})$ defined by

$$\begin{pmatrix} A' & B \\ 0 & A'' \end{pmatrix} \mapsto (A', A''),$$

gives a unique decomposition $\tilde{\xi} = (\xi', \xi'')$ where ξ' is a $GL(r, \mathbb{C})$-bundle and ξ'' is a $GL(q-r, \mathbb{C})$-bundle. We call ξ' a *subbundle of* $\tilde{\xi}$ and ξ'' a *quotient bundle of* $\tilde{\xi}$. If in general a G-bundle $\tilde{\eta}$ over P/G' which is the image of some G'-bundle $\tilde{\xi}$ under the inclusion h defined as in Theorem 3.2. Then we simply say that $\tilde{\eta}$ *has subbundle* ξ' *and quotient bundle* ξ''.

By Theorem 3.3, if $P/G' \longrightarrow M$ has a section s, then the G-bundle ξ over M is reduced to a G'-bundle $\eta = s^*\tilde{\xi}$ over M, i.e. $\xi = h\eta$. Suppose the G'-bundle $\tilde{\xi}$ is decomposed into the pair (ξ', ξ''), then $\eta = (s^*\xi', s^*\xi'')$ where $s^*\xi'$ is a $GL(r, \mathbb{C})$-bundle over M and $s^*\xi''$ is a $GL(q-r, \mathbb{C})$-bundle over M. We denote $\xi'_s = s^*\xi'$ and $\xi''_s = s^*\xi''$, then $\xi = (\xi'_s, \xi''_s)$.

(ii) Let M be a complex manifold. and $G = GL(q, \mathbb{C})$. Suppose that $W \longrightarrow M$ is a fiber bundle with structure group G and is associated to the G-bundle ξ. Let $P \longrightarrow M$ be the principal bundle with structure group (and fiber) G and be associated to ξ. Let $G' = \Delta(q, \mathbb{C})$. By the constructions in the proof of Theorem 3.2, we have the principal G'-bundle $P \xrightarrow{\tilde{\xi}} P/G'$, which is associated to $\tilde{\xi}$. The natural homomorphism $G' \longrightarrow \underbrace{\mathbb{C}^* \times \cdots \times \mathbb{C}^*}_{q\text{-times}}$ defined by

$$\begin{pmatrix} a_{11} & & \cdots & \\ 0 & a_{22} & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{qq} \end{pmatrix} \mapsto (a_{11}, a_{22}, \ldots, a_{qq})$$

gives the decomposition $\tilde{\xi} = (\xi_1, \ldots, \xi_q)$ where ξ_k is $\mathbb{C}^*$-bundle for $k = 1, \ldots q$. The ordered set $(\xi_1, \ldots, \xi_q)$ is called the *diagonal* $\mathbb{C}^*$*-bundles of* $\tilde{\xi}$. If in general

$\tilde{\eta}$ is a $GL(q,\mathbb{C})$-bundle but also the image of some $\Delta(q,\mathbb{C})$-bundle $\tilde{\xi}$ under the inclusion h defined as in Theorem 3.2. We say that *the ordered set* $(\xi_1,\dots,\xi_q)$ *are the diagonal* $\mathbb{C}^*$*-bundles of* $\tilde{\eta}$.

By Theorem 3.3, if $P/G' \longrightarrow M$ has a section s, then the G-bundle ξ over M is reduced to a G'-bundle $\eta = s^*\tilde{\xi}$ over M, i.e. $\xi = h\eta$. Suppose the G'-bundle $\tilde{\xi}$ is decomposed into the pair $(\xi_1,\dots,\xi_q)$ then $\eta = (s^*\xi_1,\dots,s^*\xi_q)$ where $s^*\xi_k$ is $\mathbb{C}^*$-bundle over M for $k=1,\dots,q$. We denote $\xi_{ks} = s^*\xi_k$ for $k=1,\dots,q$, then $\xi = (\xi_{1s},\dots,\xi_{qs})$.

3.2.4. *Decomposition of Vector Bundles.* Let $G = GL(q,\mathbb{C})$. Suppose $W \longrightarrow M$ is a fiber bundle with fiber F and structure group G. ξ is the associated G-bundle. Let P be the principal bundle with structure group and fiber G associated to the G-bundle ξ. Conversely, suppose we have P a principal G-bundle over M, associated to a G-bundle ξ, if F is a topological space that G acts on it. Then we can construct a fiber bundle $W \longrightarrow M$ with fiber F, structure group G and is associated to the same ξ-bundle in the following way: We define

$$W = P \times F/\sim$$

where $\sim$ is an equivalence relation given by $(ea,f)\sim(e,af)$ whenever $a\in G$. This may explain the name "principal".

(i) Suppose $E \longrightarrow M$ is a vector bundle with fiber $\mathbb{C}^q$. Let ξ be the associated G-bundle where $G = GL(q,\mathbb{C})$ and let P be the principal bundle with structure group and fiber G such that its fiber P_x for any $x\in M$ is the set of all isomorphisms from $\mathbb{C}^q$ to the fiber E_x. Note that P thus defined is associated with the same G-bundle ξ. Let $G' = GL(r,q-r;\mathbb{C})$ be a subgroup of G, and define the fiber bundle ${}^{[r]}E \longrightarrow M$ by ${}^{[r]}E = P/GL(r,q-r;\mathbb{C})$ as shown in the diagram

$$\begin{array}{ccc} P & \longrightarrow & {}^{[r]}E \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

Hence ${}^{[r]}E \longrightarrow M$ is the fiber bundle associated to the same G-bundle ξ as P, with structure group G and typical fiber $G/G' = G(r,q)$, the Grassmannian manifold consists of all r-planes in the complex vector space E_x for any $x\in M$.

Suppose ${}^{[r]}E \longrightarrow M$ has a section $s : M \longrightarrow {}^{[r]}E$, then this s will give a decomposition of the G-bundle $\xi = (\xi'_s,\xi''_s)$ where ξ'_s is a $GL(r,\mathbb{C})$-bundle and ξ''_s is a $GL(q-r,\mathbb{C})$-bundle. Simultaneously, s associates to each $x\in M$ an r dimensional subspace E'_x of E_x. Define

$$E' = \cup_{x\in M} E'_x,$$

then this is a vector bundle over M of rank r with typical fiber $\mathbb{C}^r$ and is associated to the $GL(r,\mathbb{C})$-bundle ξ'_s. Define

$$E'' = \cup_{x\in M} E_x/E'_x$$

then this is a vector bundle over M of rank $q-r$ with typical fiber $\mathbb{C}^{q-r}$ and is associated to the $GL(q-r,\mathbb{C})$-bundle ξ''_s. Hence we have decompose (E,ξ) into (E',ξ'_s) and (E'',ξ''_s) by using the group $G(r,q-r;\mathbb{C})$.

REMARK 3.1. The existence of s depends on the decomposability of E itself.

(ii) Let $E \longrightarrow M$ be a vector bundle with typical fiber $\mathbb{C}^q$ and associated to the G-bundle ξ where $G = GL(q;\mathbb{C})$. Let $P \longrightarrow M$ be the principal bundle with fiber and structure group G and associated to ξ. Let $G' = \Delta(q,\mathbb{C})$ be a subgroup of $GL(q,\mathbb{C})$ and define a fiber bundle ${}^{\Delta}E \longrightarrow M$ by ${}^{\Delta}E = P/\Delta(q,\mathbb{C})$, as shown in the diagram:

$$\begin{array}{ccc} P & \longrightarrow & {}^{\Delta}E \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

Hence ${}^{\Delta}E$ is the fiber bundle associated to the same G-bundle ξ as P with structure group G and typical fiber $F(q)$, the manifold of flags. Note that each fiber is a manifold of flags and each flag in the fiber ${}^{\Delta}E_x$ can be written as the following sequence

$$\{0\} =_x L_0 \subset_x L_1 \subset \cdots \subset_x L_q = {}^{\Delta}E_x$$

with $dim_x L_r = r, r = 0,\ldots,q$.

Suppose ${}^{\Delta}E \longrightarrow M$ has a section $s : M \longrightarrow {}^{\Delta}E$, then this s will give a decomposition of the G-bundle $\xi = (\xi_{1s},\ldots,\xi_{qs})$ where ξ_{ks} is $\mathbb{C}^*$-bundle for $k = 1,\ldots,q$. Simultaneously, s associates to each $x \in M$ a flag

$$s(x) = (\{0\} =_x L_0 \subset_x L_1 \subset \cdots \subset_x L_q = {}^{\Delta}E_x).$$

For each $r = 0,\ldots,q$, define

$$E_{(r)} = \cup_{x\in M\,x} L_r.$$

This is a vector bundle over M with typical fiber $\mathbb{C}^r$ and further we have the exact sequence of vector bundles over M for all $r = 0,\ldots,q$:

$$0 \longrightarrow E_{(r)} \longrightarrow E_{(r+1)} \longrightarrow A_{r+1} \longrightarrow 0$$

where $A_1 = E_{(1)}, A_r = E_{(r+1)}/E_{(r)} = \cup_{x\in M\,x} L_{r+1}/_x L_r, r = 2,\ldots,q$ such that A_k is associated to the $\mathbb{C}^*$-bundle ξ_{ks} for $k = 1,\ldots,q$. We call $A_1,\ldots,A_q$ the *diagonal line bundles* determined by the section s.

3.3. Chern Classes in the Form of Čech Cohomology.

3.3.1. *Understanding Grassmannian Manifolds and Universal Bundles.* We will first give the geometric meaning of the Grassmannian manifolds. Consider a curve $c(t), t \in [0,1]$ in $\mathbb{R}^{m+1}$ for some $m \geq 1$, the Gauss map

$$\tau : C \longrightarrow S^m$$

sends each point $c(t)$ in the curve to its unit tangent vector $c'(t) \in S^m \subset \mathbb{R}^{m+1}$. If we disregard the orientation, the Gauss map is generalized to a map, still denoted as τ,

$$\tau : c(t) \longrightarrow \mathbb{P}^m_{\mathbb{R}}$$

by projectivizing each unit tangent vector $c'(t)$ into a point in $\mathbb{P}^m_{\mathbb{R}}$. Generalize the case to a surface $S = \{s(x,y) : (x,y) \in [0,1] \times [0,1]\}$ in $\mathbb{R}^m$, $m \geq 2$, then the generalized Gauss map

$$\tau : S \longrightarrow G(2, m; \mathbb{R})$$

will send each point $s(x,y)$ to its tangent plane $T_{s(x,y)}S$, which is a two dimensional subspace in $\mathbb{R}^m$ and thus a point in the Grassmannian manifold $G(2, m; \mathbb{R})$. This can be generalized further. Suppose M is a complex manifold of complex dimension r lying in the space $\mathbb{C}^{m+1}$ for some $m \geq r$, then the generalized Gauss map

$$\tau : M \longrightarrow G(r, m)$$

will send each point $x \in M$ to the tangent space T_xM which is a point in the complex Grassmannian manifold $G(r, m)$. Finally, consider a complex vector bundle of rank r over a complex manifold M, we may also get a generalized Gauss map that maps each point $x \in M$ to its fiber E_x, which is a complex r-dimensional vector space. However, we will not know the definite dimension of the Grassmannian manifold which is needed to put every such E_x in. But we can always choose a sufficiently large dimension m. Hence we have the generalized Gauss map

$$\tau : M \longrightarrow G(r, m)$$

for m sufficiently large.

Definition 3.1 (Universal Bundle of Grassmannian Manifold). Let $G(r, m)$ be a complex Grassmannian manifold, the subbundle

$$U(r, m) \longrightarrow G(r, m)$$

of the trivial bundle defined by

$$U(r, m) = \{(P, v_P) \in G(r, m) \times \mathbb{C}^r : v_P \in P\}$$

is called the *universal bundle of the Grassmannian manifold*. In particular, if the Grassmannian manifold is a projective space, then the bundle defined is called the *tautological line bundle of the projective space*.

Consider the induced bundle as shown in the diagram

$$\begin{array}{ccc} E & \xleftarrow{\tau^*} & U(r,m) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\tau} & G(r,m). \end{array}$$

Intuitively, the image of Gauss map $\tau : M \longrightarrow G(r, m)$ gives all the information about the vector bundle E, in particular, where each r-plane comes from. The process of defining universal bundle of a Grassmannian manifold is tautological. Hence the universal bundle should have the same information as the base

Grassmannian manifold. If we canonically pull back the universal bundle to a bundle over M, it is possible to get the original vector bundle E up to an isomorphism. This may explain the name of "universal bundle". There is a proof of this theorem when the vector bundle is tangent bundle in [Milnor] P61.

THEOREM 3.4. *The vector bundle $E \longrightarrow M$ is isomorphic to the induced bundle over M by the map $\tau : M \longrightarrow G(r,m)$ from the universal bundle of the Grassmannian manifold, $\tau^*U(r,m) \longrightarrow M$. Furthermore, any map $f : M \longrightarrow G(r,m)$ homotopy to τ gives the same induced bundle up to an isomorphism.*

REMARK 3.2. This is exactly Theorem 3.1.

3.3.2. *Chern Classes.* Let M be an n-dimensional complex manifold, $E \longrightarrow M$ be a vector bundle of rank r and ξ be the associated $GL(r,\mathbb{C})$-bundle. Let $P \longrightarrow M$ be the principal bundle associated with ξ. By Theorem 3.4, to define Chern classes of ξ it suffices to define Chern classes of universal bundle of the Grassmannian manifold. We will first consider the case when E is of rank 1 and hence the Grassmannian manifold is a projective space. Then by the decomposition of fiber bundles, we can define the Chern classes in the rank q cases.

Step 1. Let ξ be a $\mathbb{C}^*$-bundle over M. Suppose the generalized Gauss map is

$$\tau : M \longrightarrow \mathbb{P}^m_{\mathbb{C}}.$$

Suppose $\mathbb{P}^m_{\mathbb{C}}$ has the homogeneous coordinates $[z_0, z_1, \ldots, z_m]$ and open covering $\{U_i\}_{i\in I}$ defined by $U_i = \{[z_0, \ldots, z_m] : z_i \neq 0\}$ for $i \in I$. Denote $h_{ij} = \frac{z_i}{z_j}$ the transition function of the tautological line bundle $U(1,m)$ over $\mathbb{P}^m_{\mathbb{C}}$. Denote the associated $\mathbb{C}^*$-bundle by $\eta \in \check{\mathrm{H}}(\mathbb{P}^m_{\mathbb{C}}, \mathbb{C}^*_\omega)$. Let $\mathbb{P}^{m-1}_{\mathbb{C}}$ denote the submanifold defined by the function $z_0 = 0$. Then it defines an singular homology class $[\mathbb{P}^{m-1}_{\mathbb{C}}]$ in $H_{2m-2}(\mathbb{P}^m_{\mathbb{C}}, \mathbb{Z})$. Let $h \in \check{\mathrm{H}}^2(\mathbb{P}^m_{\mathbb{C}}, \mathbb{Z})$ denote the Poincaré dual of $[\mathbb{P}^{m-1}_{\mathbb{C}}]$. Then the *first Chern classes* $c_1(U(1,m))$ of the tautological line bundle of $\mathbb{P}^m_{\mathbb{C}}$ is defined to be h. Equivalently, if we consider the following short exact sequence of sheaves over $\mathbb{P}^m_{\mathbb{C}}$:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C}_\omega \overset{exp(2\pi i\cdot)}{\longrightarrow} \mathbb{C}^*_\omega \longrightarrow 0,$$

then the connection operator δ in the induced long exact sequence

$$\cdots \longrightarrow H^1(\mathbb{P}^m_{\mathbb{C}}, \mathbb{C}^*_\omega) \overset{\delta}{\longrightarrow} H^2(\mathbb{P}^m_{\mathbb{C}}, \mathbb{Z}) \longrightarrow \cdots$$

will give $\delta[\eta] = h$. The *first Chern class* of the $\mathbb{C}^*$-bundle ξ over M is defined by $\tau^*h \in H^2(M, \mathbb{Z})$. The well-definedness is guaranteed by Theorem 3.4.

Step 2. Let ξ now be a $GL(q,\mathbb{C})$-bundle over M and P be the principal bundle with structure group and fiber $G = GL(q,\mathbb{C})$ associated with ξ. Let $G' = \Delta(q,\mathbb{C})$. Consider the following commutative diagram,

$$\begin{array}{ccc} P & \overset{\sigma}{\longrightarrow} & P/G' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\rho} & \\ & M & \end{array}$$

By the method of decomposition of fiber bundles, the induced G-bundle $\rho^*\xi$ over P/G' can be decomposed into q $\mathbb{C}^*$-bundles $\xi_1, \dots, \xi_q$ over P/G'. By step 1, we are able to define the first Chern class for $\mathbb{C}^*$-bundle over a complex manifold. Hence denote the first Chern class of ξ_i over the manifold P/G' by $c_1(\xi_i) = \gamma_i \in H^2(P/G', \mathbb{Z})$. Define the *total Chern class* of $\rho^*\xi = (\xi_1, \dots, \xi_q)$ over P/G' by

$$c(\rho^*\xi) = c(\xi_1) \cdot \dots \cdot c(\xi_q) = \Pi_{i=1}^q (1 + \gamma_i) \in H^{2*}(P/G', \mathbb{Z}).$$

Define the *total Chern class* $c(\xi)$ of ξ over M as a cohomology class in $H^{2*}(M, \mathbb{Z})$ by the identity

$$\rho^* c(\xi) = c(\rho^*\xi).$$

Define the *j-th Chern class* $c_j(\xi)$ of ξ to be the term of degree j in the polynomial of the total Chern class $c(\xi)$. This definition is well defined only when the $\gamma_i's$ are the image of some $\Gamma_i's$ in $H^2(M, \mathbb{Z})$ under the mapping ρ^*. Borel's fundamental theorem can guarantee this. See [Hir] p. 61.

The *Chern classes* of the vector bundle E with fiber $\mathbb{C}^r$ and associated $GL(r, \mathbb{C})$-bundle ξ are defined to be the Chern classes of ξ. The *Chern classes* of the complex manifold M is defined to be the Chern classes of the tangent bundle TM of M. The above definition can be summarized to an axiomatic definition of Chern classes whose uniqueness is shown in [Hir] p. 59.

DEFINITION 3.2 (Chern Classes).

(Axiom I) For every continuous $U(q)$-bundle ξ over a complex manifold M and every integer i , $0 \le i \le q$, there is a Chern class $c_i(\xi) \in H^{2i}(X, \mathbb{Z})$ and $c_0 = 1$ is the unit element. The total Chern class defined by

$$c(\xi) = \sum_{i=0}^{q} c_i(\xi)$$

(Axiom II) A continuous map $f : Y \longrightarrow X$ induces a map $f^* : H^1(X, U(q)_c) \longrightarrow H^1(Y, U(q)_c)$ and a homomorphism $f^* : H^*(X, \mathbb{Z}) \longrightarrow H^*(Y, \mathbb{Z})$ then

$$c(f^*\xi) = f^*(c(\xi)).$$

(Axiom III) If $\xi_1, \dots, \xi_q$ are continuous $U(1)$-bundles then

$$c(\xi_1 \oplus \dots \oplus \xi_q) = c(\xi_1) \cdot \dots \cdot c(\xi_q).$$

(Axiom IV) With the notations in the definition by Čech cohomology,

$$c(\eta) = 1 + h.$$

3.4. Chern Classes in the Form of the de Rham Cohomology. S. S. Chern pointed out that Chern classes can be represented by curvature form since both of them measure how far the vector bundle is away from a trivial product structure. (See [Chern1]). We will in the next section give the definition of Chern classes by curvature form by the language of de Rham cohomology theory. To find invariants of the vector bundle such that on the one hand represented by the curvature form and on the other hand of more than degree 2 forms. The notion of invariant polynomial meets the two needs.

3.4.1. *De Rham Cohomology of Vector Bundles.* Suppose E is a vector bundle over a complex manifold M and $A(E)$ denote the set of all sections of E over M, then the *de Rham cohomology group of the vector bundle* E is defined by the de Rham cohomology group of the complex manifold M with coefficients in $A(E)$. Denote as $H^*_{DR}(E)$. Since the exterior differential operator d leaves fixed the coefficient in $A(E)$ of a differential form, all the properties of the de Rham cohomology group of the complex manifold M remain true.

3.4.2. *Chern Classes.* Let $E \longrightarrow M$ be a complex vector bundle of rank r with metric h. Under the frame e chosen for E, the connection matrix is θ_e and the curvature matrix is Θ_e.

Definition 3.3 (Invariant Polynomial). Let $A = (A_{ij})_{r\times r}$ be a matrix, a homogeneous polynomial defined by $P(A) = P(A_{11}, \ldots, A_{ij}, \ldots, A_{rr})$ is called *invariant* if

$$P(A) = P(gAg^{-1})$$

for any $g \in GL(r, \mathbb{C})$.

Since the wedge product commutes on even degree differential forms, then an invariant polynomial $P(\Theta_e)$ gives a $End(E)$-valued even degree differential form on M which is independent of the frame e chosen, since the change of curvature matrices with respect to the change of frame is given by $\Theta'_e = g\Theta_e g^{-1}$ for $e' = ge$. In the following context we consider the invariant polynomial $P(A) = \sigma_k(A)$ where σ_k is the coefficient of t^k in the polynomial

$$det(I + tA) = \sum_{k=0}^{n} \sigma_k(A)t^k.$$

$\sigma_k(A)$ for $k = 1, \ldots, n$ is an invariant polynomial. The following lemma is from [Madsen].

Lemma 3.3. *(i) The invariant polynomial $P(\Theta)$ is a closed form and hence defines a cohomology class $[P(\Theta)] \in H^*_{DR}(E)$.*

(ii) The cohomology class $[P(\Theta)]$ is independent of choices of connections.

Proof. Before proving (i), we need two facts: Fact 1. (The Bianchi identity)

$$d\Theta_e = \theta_e \wedge \Theta_e - \Theta_e \wedge \theta_e$$

under a frame e. Indeed, by differentiating the Cartan structure equation $\Theta_e = \theta_e - \theta_e \wedge \theta_e$, we have

$$d\Theta_e = -D\theta_e \wedge \theta_e + \theta_e \wedge D\theta_e = -\Theta_e \wedge \theta_e + \theta_e \wedge \Theta_e.$$

Fact 2.

$$P'(\Theta_e) \wedge \Theta_e = \Theta_e \wedge P'(\Theta_e)$$

where $P'(\Theta_e) = \left(\frac{\partial P(\Theta_e)}{\partial \Theta_{ij}}\right)^t$. This can be shown by applying $\frac{d}{dt}$ on

$$P((I + tE_{ij})\Theta_e) = P(\Theta_e(I + tE_{ij}))$$

where E_{ij} is the n by n matrix with the (i,j)-th entry 1 and other entries all zero.

(i)

$$\begin{aligned}
dP(\Theta_e) &= \sum_{i,j} \frac{\partial P(\Theta_e)}{\partial \Theta_{ij}} \wedge d\Theta_{ij} \\
&= Tr(P'(\Theta_e) \wedge d\Theta_e) \\
&= Tr(P'(\Theta_e) \wedge \theta_e \wedge \Theta_e - \Theta_e \wedge \theta_e) \\
&= Tr(P'(\Theta_e) \wedge \theta_e \wedge \Theta_e - P'(\Theta_e) \wedge \Theta_e \wedge \theta_e) \\
&= Tr(\Theta_e \wedge P'(\Theta_e) \wedge \theta_e - P'(\Theta_e) \wedge \Theta_e \wedge \theta_e) \\
&= \Theta_e \wedge P'(\Theta_e) = 0
\end{aligned}$$

(ii) Let D_1 and D_2 be two connections of E with matrix θ_1 and θ_2 under a frame e of E. Define a projection $\pi : M \times \mathbb{R} \longrightarrow M$ and get the induced bundle map from $\pi^* E \longrightarrow \pi^* E$. Let $\pi^* D_1 = \tilde{D}_1$ and $\pi^* D_2 = \tilde{D}_2$ be the induced connection on the induced vector bundle $\pi^* E \longrightarrow M \times \mathbb{R}$. Define a connections $\tilde{D}(t)$ on $\pi^* E$ by

$$\tilde{D}(t) = (1-t)\tilde{D}_1 + t\tilde{D}_2$$

at each $(p,t) \in M \times \mathbb{R}$. If $i_0 : M \longrightarrow M \times \mathbb{R}$ and $i_1 : M \longrightarrow M \times \mathbb{R}$ denote the inclusion on the bottom and on the top respectively, we have

$$i_0^* \tilde{D} = D_1; \quad i_1^* \tilde{D} = D_2$$

and also for there corresponding curvature matrices,

$$i_0^* \tilde{\Theta} = \Theta_1; \quad i_1^* \tilde{\Theta} = \Theta_2$$

and hence

$$i_0^* (P(\tilde{\theta})) = P(\Theta_1); \quad i_1^* P(\tilde{\Theta}) = P(\Theta_2)$$

Finally since i_0 is homotopy to i_1 we have $i_0^*([P(\tilde{\theta})]) = i_1^*([P(\tilde{\theta})])$ and hence $[P(\Theta_1)] = [P(\Theta_2)]$. □

Definition 3.4 (Chern Forms, Chern Classes). Define the *Chern forms* $c_i(\Theta)$ of the curvature Θ of E by

$$c_i(\Theta) = \sigma_i \left(\frac{\sqrt{-1}}{2\pi} \Theta \right), \quad i = 0, \ldots, r.$$

Define the *i-th Chern classes of E* by

$$c_i(E) = \left[\sigma_i \left(\frac{\sqrt{-1}}{2\pi} \Theta \right) \right] \in H^{2i}_{DR}(E), \quad i = 1, \ldots, r.$$

For M a complex manifold of dimension n, define the *i-th Chern classes $c_i(M)$ of M* to be the i-th Chern classes of its tangent bundle TM for $i = 1, \ldots, n$.

(i) of Lemma 3.3 guarantees the existence of the definition of Chern classes and (ii) of Lemma 3.3 guarantees the well-definedness of the definition. That is independent choices of connections and hence of curvatures.

DEFINITION 3.5 (Chern Number). The *Chern number* $c_{i_1}c_{i_2}\dots c_{i_k}$ with $i_1+\cdots+i_k=m\le n$ of E is defined by

$$c_{i_1}c_{i_2}\dots c_{i_k}=\int_M c_{i_1}(\Theta)c_{i_2}(\Theta)\dots c_{i_k}(\Theta)\omega^{n-m},$$

where ω is the Kähler form of M.

3.4.3. *First and Second Chern Classes.* For later use, the subsection gives explicit expressions of the first and the second Chern classes.

(i) By Definition 3.4, the Chern classes are

$$c_1(E)=\left[\sigma_1\left(\frac{\sqrt{-1}}{2\pi}\Theta\right)\right];\quad c_2(E)=\left[\sigma_2\left(\frac{\sqrt{-1}}{2\pi}\Theta\right)\right].$$

The Chern forms are

$$c_1(\Theta)=\sigma_1\left(\frac{\sqrt{-1}}{2\pi}\Theta\right)=\left(\frac{\sqrt{-1}}{2\pi}\right)\text{Trace}(\Theta)=\left(\frac{\sqrt{-1}}{2\pi}\right)\left(\sum_\alpha {\Theta_\alpha}^\alpha\right);$$

$$\begin{aligned}c_2(\Theta)&=\frac{-1}{4\pi^2}\sigma_2(\Theta)=\frac{-1}{4\pi^2}\sum_{|I|=2}det(\Theta_{I,I})\\&=\frac{-1}{4\pi^2}\cdot\frac{1}{2}\cdot\sum_{\alpha,\beta}{\Theta_\alpha}^\alpha\wedge{\Theta_\beta}^\beta-{\Theta_\alpha}^\beta\wedge{\Theta_\beta}^\alpha,\end{aligned}$$

where $|I|$ stands for the number of indices in I.

(ii) Suppose E is a Hermitian vector bundle. Recall that we have ${\Theta_\alpha}^\beta$'s are $End(E)$-valued $(1,1)$-forms on M and ${\Theta_\alpha}^\beta=\sum_{k,l}{\Theta_\alpha}^\beta{}_{k\bar{l}}dz^k\wedge d\bar{z}^l$ where ${\Theta_\alpha}^\beta{}_{k\bar{l}}={\Theta_\alpha}^\beta\left(\frac{\partial}{\partial z_k},\frac{\partial}{\partial\bar{z}_l}\right)$. Then the first and second Chern forms are given by:

$$c_1(\Theta)=\frac{\sqrt{-1}}{2\pi}\sum_\alpha{\Theta_\alpha}^\alpha=\frac{\sqrt{-1}}{2\pi}\sum_\alpha\sum_{k,l}{\Theta_\alpha}^\alpha{}_{k\bar{l}}dz^k\wedge d\bar{z}^l;$$

$$\begin{aligned}c_2(\Theta)&=\frac{-1}{4\pi^2}\cdot\frac{1}{2}\sum_{\alpha,\beta}{\Theta_\alpha}^\alpha\wedge{\Theta_\beta}^\beta-{\Theta_\alpha}^\beta\wedge{\Theta_\beta}^\alpha\\&=\frac{-1}{8\pi^2}\sum_{\alpha,\beta}\sum_{p,q,r,s}\left({\Theta_\alpha}^\alpha{}_{p\bar{q}}{\Theta_\beta}^\beta{}_{r\bar{s}}-{\Theta_\alpha}^\beta{}_{p\bar{q}}{\Theta_\beta}^\alpha{}_{r\bar{s}}\right)dz^p\wedge d\bar{z}^q\wedge dz^r\wedge d\bar{z}^s.\end{aligned}$$

(iii) Suppose M is a Hermitian manifold and E is the tangent bundle TM, then the first and second Chern forms are given by:

$$c_1(\Theta)=\frac{\sqrt{-1}}{2\pi}\sum_i{\Theta_i}^i=\frac{\sqrt{-1}}{2\pi}\sum_i\sum_{k,l}{\Theta_i}^i{}_{k\bar{l}}dz^k\wedge d\bar{z}^l;\tag{27}$$

$$c_2(\Theta)=\frac{-1}{4\pi^2}\cdot\frac{1}{2}\sum_{i,j}{\Theta_i}^i\wedge{\Theta_j}^j-{\Theta_i}^j\wedge{\Theta_j}^i\tag{28}$$

$$= \frac{-1}{8\pi^2} \sum_{i,j} \sum_{p,q,r,s} \left(\Theta_i{}^i{}_{p\bar{q}} \Theta_j{}^j{}_{r\bar{s}} - \Theta_i{}^j{}_{p\bar{q}} \Theta_j{}^i{}_{r\bar{s}} \right)$$
$$dz^p \wedge d\bar{z}^q \wedge dz^r \wedge d\bar{z}^s.$$

CHAPTER 3

Chern-Number Inequalities on Compact Kähler Surfaces

Four inequalities of Chern numbers will be given in this chapter. The first three are on Kähler surfaces and the fourth one is on stable vector bundles over compact Kähler manifolds. The topics and the proofs are suggested by Professor Ngaiming Mok.

1. Chern-Number Inequality on Compact Kähler Surfaces

We will first give a background on the Chern-number inequality $c_1^2 \leq 3c_2$ on a compact Kähler surface. Then by using the invariant theory get a specific constant in an identity representing the polynomial $c_1^2 - 3c_2$ by holomorphic sectional curvatures under the assumption that M is endowed with a Kähler-Einstein metric whose existence is given by the solution of Calabi's conjecture. Thirdly, we will represent the proof of a known result, see for example [Yau]: $c_1^2 = 3c_2$ under the assumption of $c_1^2 < 0$ i.e. negative Einstein constant, if and only if M is biholomorphic to a ball with Bergman metric. Finally, we will give the relation of $c_1^2 - 3c_2$ and the polynomial in terms of holomorphic sectional curvature without using the condition of being Einstein.

1.1. Background.

The references of this subsection are [Besse] and [Yau].

Definition 1.1 (Einstein Manifold). A Hermitian manifold M with Hermitian metric h, Hermitian connection D and curvature Θ is said to be an *Einstein manifold* if its Ricci form is equals to its Kähler form, i.e. the fundamental (1, 1)-form of M up to a constant. That is

$$\frac{\sqrt{-1}}{2\pi}\sum_{i,j} Ric_{i\bar{j}}dz^i \wedge d\bar{z}^j = \lambda\sqrt{-1}\sum_{i,j} h_{ij}dz^i \wedge d\bar{z}^j.$$

Or equivalently, the Ricci curvature tensor is equal to the metric tensor up to a constant. That is,

$$Ric_{i\bar{j}} = \lambda h_{ij}.$$

The condition of a manifold to be *Einstein* is weaker than the condition to be of constant sectional curvature, which makes the class of Einstein manifolds larger than the class of Kähler manifolds with constant holomorphic sectional curvature. The solution of *Calabi's conjecture* gives a large class of Kähler-Einstein manifolds: Let M be a compact Kähler manifold. Recall (21) and the expression (27), we have the first Chern form of M is the Ricci form of the tangent bundle of M:

$$c_1(\Theta) = \frac{\sqrt{-1}}{2\pi} \sum_{k,l} Ric_{k\bar{l}} dz^k \wedge d\bar{z}^l.$$

That is, the Ricci form is in the cohomology class $c_1(M)$. If M is further Einstein, then the Kähler form which equals to the Ricci form up to a constant, will also in the class $c_1(M)$. This gives an obstruction for a compact Kähler manifold to be Einstein. Calabi conjectured that this was the only obstruction:

THEOREM 1.1 (Calabi's Conjecture). *Let M be a compact Kähler manifold and $c_1(M)$ is its first Chern class. Then any closed real $(1,1)$-form in the class $c_1(M)$ is the Ricci form (or Kähler form) of a unique Kähler-Einstein metric.*

Yau proved the Calabi's conjecture in [Yau]. As an corollary, we have (cited in [Besse] p. 325.)

THEOREM 1.2. *Let M be a compact Kähler manifold of complex dimension n with negative first Chern number. Then*

$$(-1)^n (2(n+1) c_2 c_1^{n-2} - n{c_1}^n) \geq 0.$$

Equality holds if and only if the holomorphic sectional curvature of the induced Kähler-Einstein metric is constant negative so that M is covered by the unit ball $\mathbb{C}^n$.

1.2. Inequality $c_1^2 \leq 3c_2$ on Compact Kähler Surfaces. The main objective of this subsection is to understand Theorem 1.2 in the dimension two case in the angle of the invariant theory. By Theorem 1.1, the compact Kähler surface M can be endowed with a Kähler-Einstein metric, call it g. Fix this metric as the metric of M and let D be the Hermitian connection, Θ be the curvature in the following context. There is a known formula in the invariant theory:

$$c_1^2 - 3c_2 = G \int_{\alpha \in S_x^3 \subset T_x(M)} (R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} - \text{Average} R_{\alpha\bar{\alpha}\alpha\bar{\alpha}})^2 dV(M,g), \tag{29}$$

on any point x of a compact Kähler-Einstein surface (M, g) with the constant G being negative. This gives the inequality $c_1^2 \leq 3c_2$ on any compact Kähler surface. We will compute the constant G in the formula (29). There is a fact which we will assume from the invariant theory (refer to [Besse] p. 133):

THEOREM 1.3. *Any polynomial of degree 2 in the curvature tensors which is invariant under the unitary transformation can be represented in three basis with coefficients in $\mathbb{R}$:*

$$\left\{ \sum_{i,j,k,l} |R_{i\bar{j}k\bar{l}}|^2, \quad \sum_{i,j} |Ric_{ij}|^2, \quad \sum_{i,j,k,l} R_{i\bar{i}j\bar{j}} R_{k\bar{k}l\bar{l}} \right\}. \tag{30}$$

Fix any $x \in M$, the following polynomial

$$\int_{\alpha \in S_x^3 \subset T_x(M)} (R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} - \text{Average} R_{\alpha\bar{\alpha}\alpha\bar{\alpha}})^2 dV(M,g) \tag{31}$$

is a degree two polynomial in the curvature tensors which is invariant under the unitary transformation. By Theorem 1.3, (31) can be represented uniquely by the basis in (30). On the other hand, the Chern numbers can be represented by curvature tensors. Hence the constant G can be calculated by methods of indeterminate coefficients.

1.2.1. *Calculation of* $\int_{\alpha \in S_x^3 \subset T_x(M)} (R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} - \mathit{Average} R_{\alpha\bar{\alpha}\alpha\bar{\alpha}})^2 dV(M,g)$. We will calculate the integration (31)

$$\int_{\alpha \in S_x^3 \subset T_x(M)} (R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} - \text{Average} R_{\alpha\bar{\alpha}\alpha\bar{\alpha}})^2 dV(M,g)$$

term by term. For each term, we can use the condition of being Einstein to reduce into only two coefficients. We will get the coefficients by evaluating at two examples, say $\mathbb{P}^2_{\mathbb{C}}$ and $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$.

Step 1. Compute $\int_{\alpha \in S_x^3} R^2_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$: Since $\int_{\alpha \in S_x^3} R^2_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$ is invariant under unitary transformation, we have

$$\int_{\alpha \in S_x^3} R^2_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = A \sum_{i,j,k,l} |R_{i\bar{j}k\bar{l}}|^2 + B \sum_{i,j} |Ric_{i\bar{j}}|^2 + C \sum_{i,j,k,l} R_{i\bar{i}j\bar{j}} R_{k\bar{k}l\bar{l}}, \tag{32}$$

for some constants A, B and C. Use the condition of Kähler-Einstein, and choose the local coordinates such that $g_{i\bar{j}}(x) = \delta_{ij}(x)$, then

$$\sum_{i,j} |Ric_{i\bar{j}}|^2 = \sum_{i,j} \lambda^2 |g_{i\bar{j}}|^2 = \lambda^2 n,$$

$$\sum_{i,j,k,l} R_{i\bar{i}j\bar{j}} R_{k\bar{k}l\bar{l}} = \sum_{i,j} Ric_{i\bar{i}} Ric_{j\bar{j}} = \lambda^2 \sum_{i,j} g_{i\bar{i}} g_{j\bar{j}} = \lambda^2 n^2.$$

Thus,

$$\sum_{i,j} |Ric_{i\bar{j}}|^2 = \frac{1}{n} \sum_{i,j,k,l} R_{i\bar{i}j\bar{j}} R_{k\bar{k}l\bar{l}}.$$

Write

$$\int R^2_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = A \sum_{i,j,k,l} |R_{i\bar{j}k\bar{l}}|^2 + E \sum_{i,j,k,l} R_{i\bar{i}j\bar{j}} R_{k\bar{k}l\bar{l}} \tag{33}$$

in which $E = \frac{B}{n} + C$ and we omit S_x^3 under the $\int$ in later context. To determine the constants A and E, we only need to consider two examples, say (a) $\mathbb{P}^2_{\mathbb{C}}$ and (b) $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$.

(a) Let $(M,g) = (\mathbb{P}^2_{\mathbb{C}}, g_{FS})$ where $g_{FS} = \sum g_{i\bar{j}} dz^i \otimes d\bar{z}^j$ is the Fubini-Study metric defined in (3): Without loss of generality, we consider the point $x \in \mathbb{P}^2_{\mathbb{C}}$ and such that the first coordinate of x is non-zero and choose special coordinates

$[1, z_1, z_2]$ such that $g_{i\bar{j}}(x) = \delta_{ij}, dg_{i\bar{j}}(x) = 0$ whose existence is proved in [Mok] (p. 26). By (23), we have at the point x:

$$R_{i\bar{j}k\bar{l}} = g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}} = \delta_{ij}\delta_{kl} + \delta_{il}\delta_{kl}$$

where $i, j, k, l = 1, 2$. Therefore at x, the only non-zero curvature tensors are

$$R_{1\bar{1}1\bar{1}}, R_{1\bar{1}2\bar{2}}, R_{2\bar{1}1\bar{2}}, R_{1\bar{2}2\bar{1}}, R_{2\bar{2}1\bar{1}}, R_{2\bar{2}2\bar{2}}.$$

Use the properties: $R_{i\bar{j}k\bar{l}}R_{p\bar{q}r\bar{s}} = R_{p\bar{q}r\bar{s}}R_{i\bar{j}k\bar{l}}$, $R_{i\bar{j}k\bar{l}} = R_{i\bar{l}k\bar{j}}$ and the symmetry of z_1 and z_2. Substitute the values of $R_{i\bar{j}k\bar{l}}$ we get at x,

$$\begin{aligned}
\int R^2_{\alpha\bar{\alpha}\alpha\bar{\alpha}} &= \int 2|\alpha_1|^8 R^2_{1\bar{1}1\bar{1}} + 2|\alpha_1|^6|\alpha_2|^2(2R_{1\bar{1}1\bar{1}}R_{1\bar{1}2\bar{2}} + 2R_{1\bar{1}1\bar{1}}R_{2\bar{1}1\bar{2}} \\
&\quad +2R_{1\bar{1}1\bar{1}}R_{1\bar{1}1\bar{1}}R_{1\bar{2}2\bar{1}} + R_{1\bar{1}1\bar{1}}R_{2\bar{2}1\bar{1}}) \\
&\quad +|\alpha_1|^4|\alpha_2|^4(2R^2_{1\bar{1}2\bar{2}} + 2R^2_{2\bar{1}1\bar{2}} + 2R_{1\bar{1}2\bar{2}}R_{2\bar{1}1\bar{2}} + 2R_{1\bar{1}2\bar{2}}R_{1\bar{2}2\bar{1}} \\
&\quad +2R_{1\bar{1}2\bar{2}}R_{2\bar{2}1\bar{1}} + 2R_{2\bar{1}1\bar{2}}R_{1\bar{2}2\bar{1}} + 2R_{2\bar{1}1\bar{2}}R_{2\bar{2}1\bar{1}} + 2R_{1\bar{2}2\bar{1}}R_{2\bar{2}1\bar{1}}) \\
&= 8\int |\alpha_1|^8 + 32|\alpha_1|^6|\alpha_2|^2 + 24|\alpha_1|^4|\alpha_2|^4
\end{aligned}$$

To evaluate the integration in the real situation, change coordinates in the following two steps,

$$\begin{cases} \alpha_1 = x_1 + iy_1 \\ \bar{\alpha}_1 = x_1 - iy_1 \\ \alpha_2 = x_2 + iy_2 \\ \bar{\alpha}_2 = x_2 - iy_2 \end{cases}; \quad \begin{cases} x_1 = r\sin\varphi_1\sin\varphi_2\cos\theta \\ y_1 = r\sin\varphi_1\sin\varphi_2\sin\theta \\ x_2 = r\cos\varphi_1\sin\varphi_2 \\ y_2 = r\cos\varphi_2 \end{cases}$$

where $0 \leq \theta \leq 2\pi$, $0 \leq \varphi_1, \varphi_2 \leq \pi$. The determinant of the Jacobian $\dfrac{\partial(\alpha_1, \bar{\alpha}_1, \alpha_2, \bar{\alpha}_2)}{\partial(r, \theta, \varphi_1, \varphi_2)}$ is $4r^3\sin\phi_1\sin^2\varphi_2$. Thus,

$$d\alpha_1 d\bar{\alpha}_1 d\alpha_2 d\bar{\alpha}_2 = 4r^3\sin\phi_1\sin^2\varphi_2 dr d\theta d\varphi_1 d\varphi_2.$$

Evaluate the integration $\int R^2_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$: Since

$$\begin{aligned}
&\int |\alpha_1|^8 d\alpha_1 d\bar{\alpha}_1 d\alpha_2 d\bar{\alpha}_2 \\
= &\int_{S^3} \sin^8\varphi_1\sin^8\varphi_2 4\sin\varphi_1\sin^2\varphi_2 d\theta d\varphi_1 d\varphi_2 \\
= &\ 8\pi\int_0^\pi \sin^9\varphi_1 d\varphi_1 \int_0^\pi \sin^{10}\varphi_2 d\varphi_2 \\
= &\ 8\pi\int_{-1}^1 (1-u^2)^4 du \int_0^\pi \left(\frac{1-\cos 2\varphi_2}{2}\right)^5 d\varphi_2 \quad (u = \cos\varphi_1) \\
= &\ 8\pi \cdot \frac{4\cdot 64}{9\cdot 35}\cdot\frac{63}{8}\pi = \frac{8}{5}\pi^2;
\end{aligned}$$

$$
\begin{aligned}
& \int |\alpha_1|^6|\alpha_2|^2 d\alpha_1 d\bar{\alpha}_1 d\alpha_2 d\bar{\alpha}_2 \\
= & \int \sin^6\varphi_1 \sin^6\varphi_2(\cos^2\varphi_1\sin^2\varphi_1 + \cos^2\varphi_2) 4\sin\varphi_1\sin^2\varphi_2 d\theta d\varphi_1 d\varphi_2 \\
= & \; 8\pi \int \cos^2\varphi_1 \sin^7\varphi_1\sin^{10}\varphi_2 + \sin^7\varphi_1\cos^2\varphi_2\sin^8\varphi_2 d\varphi_1 d\varphi_2 \\
= & \; 8\pi(\frac{1}{40}\pi + \frac{1}{40}\pi) = \frac{2}{5}\pi; \\
& \int |\alpha_1|^4|\alpha_2|^4 d\alpha_1 d\bar{\alpha}_1 d\alpha_2 d\bar{\alpha}_2 \\
= & \int_{S^3} \sin^4\varphi_1\sin^4\varphi_2(\cos^2\varphi_1\sin^2\varphi_2+\cos^2\varphi_2)^2 4\sin\varphi_1\sin^2\varphi_2 d\theta d\varphi_1 d\varphi_2 \\
= & \; 8\pi \int \cos^4\varphi_1\sin^5\varphi_1\sin^{10}\varphi_2 + 2\cos^2\varphi_1\sin^5\varphi_1\cos^2\varphi_2\sin^8\varphi_2 \\
& + \sin^5\varphi_1\cos^4\varphi_2\sin^6\varphi_2 d\varphi_1 d\varphi_2 \\
= & \; 8\pi\left(\frac{1}{80}\pi + \frac{1}{120}\pi + \frac{1}{80}\pi\right) = \frac{4}{15}\pi^2,
\end{aligned}
$$

Then we have

$$
\int R^2_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 8\cdot\frac{8}{5}\pi^2 + 32\cdot\frac{2}{5}\pi^2 + 24\cdot\frac{4}{15}\pi^2 = 32\pi^2. \tag{34}
$$

On the other hand we obtain

$$
A\sum_{i,j,k,l}|R_{i\bar{j}k\bar{l}}|^2 = A(R^2_{1\bar{1}1\bar{1}} + R^2_{1\bar{a}2\bar{2}} + R^2_{1\bar{2}2\bar{1}} + R^2_{2\bar{1}1\bar{2}} + R^2_{2\bar{2}1\bar{1}^2} + R^2_{2\bar{2}2\bar{2}}) = A\cdot 12; \tag{35}
$$

$$
E\sum_{i,j,k,l} R_{i\bar{i}j\bar{j}}\cdot R_{k\bar{k}l\bar{l}} = En^2\lambda^2 = E\cdot 36 \quad (n=2, \lambda=3). \tag{36}
$$

Substituting (34), (35) and (36) into (33), we get

$$
32\pi^2 = 12A + 36E. \tag{37}
$$

(b) Let $(M,g) = (\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}, g)$ where g is the product metric of g_{FS} of $\mathbb{P}^1_{\mathbb{C}}$: By

$$
R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = R_{1\bar{1}1\bar{1}}|\alpha_1|^4 + R_{2\bar{2}2\bar{2}}|\alpha_2|^4 = 2|\alpha_1|^4 + 2|\alpha_2|^4,
$$

we have

$$
\begin{aligned}
\int R^2_{\alpha\bar{\alpha}\alpha\bar{\alpha}} &= \int 4|\alpha_1|^8 + 8|\alpha_1|^4|\alpha_2|^4 + 4|\alpha_2|^8 \\
&= 8\int|\alpha_1|^8 + 8\int|\alpha_1|^4|\alpha_2|^4 \\
&= 8\cdot\frac{8}{5}\pi^2 + 8\cdot\frac{4}{15}\pi^2 = 8\cdot\frac{28}{15}\pi^2.
\end{aligned}
$$

That is

$$
\int R^2_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 8\cdot\frac{28}{15}\pi^2. \tag{38}
$$

On the other hand, we obtain

$$A\sum_{i,j,k,l}|R_{i\bar{j}k\bar{l}}|^2 = A(R^2_{1\bar{1}1\bar{1}} + R^2_{2\bar{2}2\bar{2}}) = A\cdot 8; \tag{39}$$

$$E\sum_{i,j,k,l}R_{i\bar{i}j\bar{j}}R_{k\bar{k}l\bar{l}} = E\cdot(R^2_{1\bar{1}1\bar{1}} + 2R_{1\bar{1}1\bar{1}}R_{2\bar{2}2\bar{2}} + R^2_{2\bar{2}2\bar{2}}) = E\cdot 16. \tag{40}$$

Substituting (38), (39) and (40) into (33), we get

$$8\cdot\frac{28}{5}\pi^2 = 8A + 16E. \tag{41}$$

By the identities (37) and (41), we get $A = \frac{4}{15}\pi^2, E = \frac{4}{5}\pi^2$. Therefore,

$$\int R^2_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = \frac{4}{15}\pi^2\sum_{i,j,k,l}|R_{i\bar{j}k\bar{l}}|^2 + \frac{4}{5}\pi^2\sum_{i,j,k,l}R_{i\bar{i}j\bar{j}}R_{k\bar{k}l\bar{l}}. \tag{42}$$

Step 2. Compute Average$R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$: Again by invariant theory there exists some constant D such that

$$\text{Average}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = D\sum_{i,j}R_{i\bar{i}j\bar{j}} \tag{43}$$

at any point $x\in M$. To determine the constant D, we consider $(M,g) = (\mathbb{P}^2_{\mathbb{C}}, g_{FS})$, then

$$\begin{aligned}\int R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} &= \int 2R_{1\bar{1}1\bar{1}}|\alpha_1|^4 + 4R_{1\bar{1}2\bar{2}}|\alpha_1|^2|\alpha_2|^2 d\alpha_1 d\bar{\alpha}_1 d\alpha_2 d\bar{\alpha}_2\\ &= 4\cdot\frac{8}{3}\pi^2 + 4\cdot\frac{4}{3}\pi^2.\end{aligned}$$

That is

$$\int R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 16\pi^2. \tag{44}$$

Further,

$$Vol(S^3) = \int 4r^3\sin\varphi_1\sin^2\varphi_2 d\theta d\varphi_1 d\varphi_2 = 8\pi^2. \tag{45}$$

By (44) and (45), we have

$$\text{Average}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = \frac{\int R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}}{Vol(S^3)} = \frac{16\pi^2}{8\pi^2} = 2. \tag{46}$$

On the other hand we get,

$$D\sum_{i,j}R_{i\bar{i}j\bar{j}} = D(R_{1\bar{1}1\bar{1}} + R_{1\bar{1}2\bar{2}} + R_{2\bar{2}1\bar{1}} + R_{2\bar{2}2\bar{2}}) = D\cdot 6. \tag{47}$$

Substituting (46) and (47) into (43), we have $D = \frac{1}{3}$ and hence

$$\text{Average}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = \frac{1}{3}\sum_{i,j}R_{i\bar{i}j\bar{j}}. \tag{48}$$

Step 3. Compute $-2\int R_{\alpha\bar\alpha\alpha\bar\alpha}\text{Average}R_{\alpha\bar\alpha\alpha\bar\alpha}$:

$$\begin{aligned}
& -2\int R_{\alpha\bar\alpha\alpha\bar\alpha}\text{Average}R_{\alpha\bar\alpha\alpha\bar\alpha} \\
= & -2\int R_{\alpha\bar\alpha\alpha\bar\alpha}\frac{1}{3}\sum_{i,j} R_{i\bar i j\bar j} \\
= & -\frac{2}{3}\sum_{i,j} R_{i\bar i j\bar j}\cdot 8\pi^2\cdot \text{Average}R_{\alpha\bar\alpha\alpha\bar\alpha} \\
= & -\frac{2}{3}\sum_{i,j} R_{i\bar i j\bar j}\cdot 8\pi^2\cdot \frac{1}{3}\sum_{k,l} R_{k\bar k l\bar l} \\
= & -\frac{16}{9}\pi^2\sum_{i,j,k,l} R_{i\bar i j\bar j}R_{k\bar k l\bar l}.
\end{aligned}$$

That is,

$$-2\int R_{\alpha\bar\alpha\alpha\bar\alpha}\text{Average}R_{\alpha\bar\alpha\alpha\bar\alpha} = -\frac{16}{9}\pi^2\sum_{i,j,k,l} R_{i\bar i j\bar j}R_{k\bar k l\bar l}. \tag{49}$$

Step 4. Compute $\int(\text{Average}R_{\alpha\bar\alpha\alpha\bar\alpha})^2$:

$$\int(\text{Average}R_{\alpha\bar\alpha\alpha\bar\alpha})^2 = \frac{1}{9}\sum_{i,j,k,l} R_{i\bar i j\bar j}R_{k\bar k l\bar l}\cdot 8\pi^2 = \frac{8}{9}\pi^2\sum_{i,j,k,l} R_{i\bar i j\bar j}R_{k\bar k l\bar l}. \tag{50}$$

By identities (42), (49) and (50), formula (31) is as follows:

$$\begin{aligned}
& \int(R_{\alpha\bar\alpha\alpha\bar\alpha} - \text{Average}R_{\alpha\bar\alpha\alpha\bar\alpha})^2 dV(M,g) \\
= & \frac{4}{15}\pi^2\sum_{i,j,k,l}|R_{i\bar j k\bar l}|^2 + \left(\frac{4}{5}\pi^2 - \frac{16}{9}\pi^2 + \frac{8}{9}\pi^2\right)\sum_{i,j,k,l} R_{i\bar i k\bar k}R_{j\bar j l\bar l} \\
= & \frac{4}{45}\pi^2\left(-3\sum_{i,j,k,l}|R_{i\bar j k\bar l}|^2 - \sum_{i,j,k,l} R_{i\bar i j\bar j}R_{k\bar k l\bar l}\right).
\end{aligned}$$

That is,

$$\int(R_{\alpha\bar\alpha\alpha\bar\alpha} - \text{Average}R_{\alpha\bar\alpha\alpha\bar\alpha})^2 dV(M,g) = -\frac{4\pi^2}{45}\sum_{i,j,k,l}\big(3|R_{i\bar j k\bar l}|^2 + R_{i\bar i j\bar j}R_{k\bar k l\bar l}\big) \tag{51}$$

1.2.2. *Calculation of* $c_1^2 - 3c_2$. We will calculate the polynomial of Chern forms first, and then Chern numbers. By (19), and under the assumption that $h_{ij} = \delta_{ij}$, we have

$$R_{i\bar j k\bar l} = \sum_p h_{pj}\Theta_i{}^p{}_{k\bar l} = \Theta_i{}^j{}_{k\bar l}.$$

Further by (27) and (28), we have

$$\begin{aligned}
c_1^2(\Theta) - 3c_2(\Theta) &= \frac{-1}{4\pi^2}\sum_{i,j}\Theta_i{}^i \wedge \Theta_j{}^j + 3\cdot\frac{1}{8\pi^2}\sum_{i,j}(\Theta_i{}^i \wedge \Theta_j{}^j - \Theta_i{}^j \wedge \Theta_j{}^i) \\
&= \frac{1}{8\pi^2}\sum_{i,j}\Theta_i{}^i \wedge \Theta_j{}^j - \frac{3}{8\pi^2}\sum_{i,j}\Theta_i{}^j \wedge \Theta_j{}^i \\
&= \sum_{k\neq l}\Big(\frac{1}{8\pi^2}\Big(\sum_{i,j}\Theta_i{}^i{}_{k\bar{k}}\Theta_j{}^j{}_{l\bar{l}} - \Theta_i{}^i{}_{k\bar{l}}\Theta_j{}^j{}_{l\bar{k}}\Big) \\
&\quad -\frac{3}{8\pi^2}\Big(\sum_{i,j}\Theta_i{}^j{}_{k\bar{k}}\Theta_j{}^i{}_{l\bar{l}} - \Theta_i{}^j{}_{k\bar{l}}\Theta_j{}^i{}_{l\bar{k}}\Big)\Big)dz^k \wedge d\bar{z}^k \wedge dz^l \wedge d\bar{z}^l \\
&= \frac{1}{8\pi^2}\sum_{k\neq l}\Big(\sum_{i,j}R_{i\bar{i}k\bar{k}}R_{j\bar{j}l\bar{l}} - R_{i\bar{i}k\bar{l}}R_{j\bar{j}l\bar{k}} \\
&\quad -3\Big(\sum_{i,j}R_{i\bar{j}k\bar{k}}R_{j\bar{i}l\bar{l}} - R_{i\bar{j}k\bar{l}}R_{j\bar{i}l\bar{k}}\Big)\Big)dz^k \wedge d\bar{z}^k \wedge dz^l \wedge d\bar{z}^l \\
&= \frac{1}{8\pi^2}\Big(\sum_{k\neq l,i,j}R_{i\bar{i}k\bar{k}}R_{j\bar{j}l\bar{l}} + 3R_{i\bar{j}k\bar{l}}R_{j\bar{i}l\bar{k}}\Big)dz^k \wedge d\bar{z}^k \wedge dz^l \wedge d\bar{z}^l
\end{aligned}$$

Comparing with (51), we have

$$\begin{aligned}
& c_1^2(\Theta) - 3c_2(\Theta) \\
= \; & -\frac{45}{4\pi^2}\cdot\frac{1}{8\pi^2}\left(\int (R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} - \text{Average}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}})^2 dV\right) dz^k \wedge d\bar{z}^k \wedge dz^l \wedge d\bar{z}^l.
\end{aligned}$$

Integration both sides over M, we have

$$c_1^2 - 3c_2 = -\frac{45}{32\pi^4}\left(\int (R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} - \text{Average}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}})^2 dV\right)\int_M dz^k \wedge d\bar{z}^k \wedge dz^l \wedge d\bar{z}^l$$

Hence

$$c_1^2 - 3c_2 = H\left(\int (R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} - \text{Average}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}})^2 dV\right)K, \tag{52}$$

where $H = -\dfrac{45}{32\pi^4}$ and $K = \int_M dz^k \wedge d\bar{z}^k \wedge dz^l \wedge d\bar{z}^l$. Hence G in (29) equals $H \cdot K < 0$. Since the Chern numbers are independent of the metric chosen, hence we have for any compact Kähler surface, $c_1^2 \leq 3c_2$.

1.3. The Case when $c_1^2 = 3c_2$ and $c_1^2 < 0$. In this subsection we assume the first Chern number of the compact Kähler-Einstein manifold M is negative. Under the set-up in the above subsection, if M gives $c_1^2 = 3c_2$, then from formula (29), we have $R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = \text{Average}R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}$, for any $\alpha \in T_xM$ for any $x \in M$. By Proposition 2.3 and Corollary 2.5 of Chapter 2, M is of constant holomorphic sectional curvature and

$$R_{i\bar{j}k\bar{l}} = C(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}})$$

for some constant C. Observe that this constant should be negative by the assumption of negativity of first Chern number of M, hence M is of constant negative holomorphic sectional curvature. Hence under the special coordinates we have

$$R_{i\bar{j}k\bar{l}} = C(\delta_{i\bar{j}}\delta_{k\bar{l}} + \delta_{i\bar{l}}\delta_{k\bar{j}}) \quad \text{where } C < 0.$$

Hence the only non-zero curvature tensors up to symmetric transformations will be in the forms $R_{i\bar{i}j\bar{j}}, R_{i\bar{i}i\bar{i}}$, where $i, j = 1, 2$ and $i \neq j$.

LEMMA 1.1. *The curvature tensor R in this case is parallel. That is, $\nabla R = 0$.*

PROOF. By the condition that the only non-vanishing curvature tensors are $R_{i\bar{i}j\bar{j}}$ and $R_{i\bar{i}i\bar{i}}$, to show $\nabla R = 0$ it suffices to show (i) $\nabla R_{i\bar{i}j\bar{j}} = 0$, (ii) $\nabla R_{i\bar{i}i\bar{i}} = 0$. (i) $\nabla R_{i\bar{i}j\bar{j}} = 0$ if and only if $\nabla_i R_{i\bar{i}j\bar{j}} = 0$, $\nabla_j R_{i\bar{i}j\bar{j}} = 0$. By using the *Second Bianchi Identity* saying that the first and the fourth indices in the expression $\nabla_s R_{i\bar{j}k\bar{l}}$ commute, we have

$$\nabla_i R_{i\bar{i}j\bar{j}} = \nabla_j R_{i\bar{i}i\bar{j}} = 0, \quad \nabla_j R_{i\bar{i}j\bar{j}} = \nabla_j R_{j\bar{j}i\bar{i}} = \nabla_i R_{j\bar{j}j\bar{i}} = 0.$$

Hence $\nabla R_{i\bar{i}j\bar{j}} = 0$.

(ii) $\nabla R_{i\bar{i}i\bar{i}} = 0$ if and only if $\nabla_i R_{i\bar{i}i\bar{i}} = 0$, $\nabla_j R_{i\bar{i}i\bar{i}} = 0$. Since for any $\alpha, \beta \in T_x^{1,0}(M)$,

$$\begin{aligned} dR_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(\beta) &= \nabla_\beta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} + R_{\nabla_\beta \alpha\bar{\alpha}\alpha\bar{\alpha}} + R_{\alpha\nabla_\beta\bar{\alpha}\alpha\bar{\alpha}} + R_{\alpha\bar{\alpha}\nabla_\beta\alpha\bar{\alpha}} + R_{\alpha\bar{\alpha}\alpha\nabla_\beta\bar{\alpha}} \\ &= \nabla_\beta R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} + R_{\nabla_\beta i\bar{j}k\bar{l}}\alpha^1\bar{\alpha}^j\alpha^k\bar{\alpha}^l + R_{i\nabla_\beta\bar{j}k\bar{l}}\alpha^i\bar{\alpha}^j\alpha^k\bar{\alpha}^l \\ &\quad + R_{i\bar{j}\nabla_\beta k\bar{l}}\alpha^1\bar{\alpha}^j\alpha^k\bar{\alpha}^l + R_{i\bar{j}k\nabla_\beta\bar{l}}\alpha^1\bar{\alpha}^j\alpha^k\bar{\alpha}^l, \end{aligned}$$

where $s, i, j, k, l = 1, 2$. Substituting $\alpha = \frac{\partial}{\partial z_i}$ and evaluating at $\beta = \frac{\partial}{\partial z_i}$, we have $dR_{i\bar{i}i\bar{i}} = \nabla_i R_{i\bar{i}i\bar{i}} + R_{\nabla_i i\bar{i}i\bar{i}} + R_{i\nabla_i\bar{i}i\bar{i}} + R_{i\bar{i}\nabla_i i\bar{i}} + R_{i\bar{i}i\nabla_i\bar{i}} = \nabla R_{i\bar{i}i\bar{i}} + 0 + 0 + 0 + 0 = 0$. By $dR_{i\bar{i}i\bar{i}} = 0$, we have $\nabla_i R_{i\bar{i}i\bar{i}} = 0$. Since

$$\nabla_j R_{i\bar{i}i\bar{i}} = \nabla_i R_{i\bar{i}j\bar{i}} = 0,$$

then $\nabla R_{i\bar{i}i\bar{i}} = 0$. Therefore $\nabla R = 0$. □

A Riemannian manifold (M, g) is said to be *Riemannian symmetric* if and only if at each point $x \in X$ there exists an involution σ_x such that x is an isolated fixed point of σ_x. There is a notion of rank of a Riemannian symmetric space: If M be a Riemannian symmetric space then M is of *rank* r if there is totally geodesic flat submanifold of real dimension r and r is the largest possible. We will state without proof the following theorem. (ref. [Kobayashi] Chapter XI)

THEOREM 1.4. *A Riemannian manifold X is Riemannian symmetric if and only if $\nabla R = 0$.*

There is known result (refer to [Yau]) as follows,

THEOREM 1.5. *Let M be a compact Kähler surface. Let g be a Kähler-Einstein metric on it. If $c_1^2 = 3c_2$ and further M is with negative first Chern number, then M will be biholomorphic to a unit ball with Bergman metric up to a constant.*

Proof. When $c_1^2 = 3c_2$, by Lemma 1.1 we have $\nabla R = 0$. By Theorem 1.4, our 2-dimensional Kähler-Einstein manifold is a Riemannian symmetric space. Since M is of negative constant holomorphic sectional curvature, then M is of negative Riemannian sectional curvature, which we assume as a fact. Hence our manifold is of rank 1. Otherwise, there would be a 2 dimensional flat submanifold which contradict to the negativity of Riemannian sectional curvature. However, rank 1 Riemannian symmetric spaces, which can be classified in terms of the Lie-algebra, is biholomorphic to a ball with Bergman metric up to a constant by classification theory. □

1.4. The Case Not Necessarily Einstein. Without using the condition of being Einstein, we will give a relation between $c_1^2 - 3c_2$ and $\int_{\alpha\in S_x^3\subset T_x(M)}(R_{\alpha\bar\alpha\alpha\bar\alpha} - \text{Average}R_{\alpha\bar\alpha\alpha\bar\alpha})^2 dV(M,g)$ appeared in the formula (29) over a compact Kähler surface. Recall for (32),

$$\int_{\alpha\in S_x^3} R^2_{\alpha\bar\alpha\alpha\bar\alpha} = A\sum_{i,j,k,l}|R_{i\bar jk\bar l}|^2 + B\sum_{i,j}|Ric_{i\bar j}|^2 + C\sum_{i,j,k,l}R_{i\bar ij\bar j}R_{k\bar kl\bar l},$$

we have obtained

$$A = \frac{4}{15}\pi^2, \quad \frac{B}{2} + C(=E) = \frac{4}{5}\pi^2. \tag{53}$$

To determine B, C, we calculate a one dimensional manifold say $\mathbb{P}^1_{\mathbb{C}}$ with Fubini-Study metric. Under homogeneous coordinates, the only non-zero curvature tensor is $R_{1\bar11\bar1}$. Therefore, for any tangent vector α in the tangent space of $\mathbb{P}^1_{\mathbb{C}}$ at a certain point, we have $R_{\alpha\bar\alpha\alpha\bar\alpha} = R_{1\bar11\bar1}|\alpha_1|^4 = -2|\alpha_1|^4$ and $R^2_{\alpha\bar\alpha\alpha\bar\alpha} = 4|\alpha_1|^8$. Change coordinate by define $\alpha_1 = x + iy$ where $x = \cos\theta$ and $y = \sin\theta$. We have $|\alpha_1| = \sqrt{x^2+y^2} = \sqrt{\cos^2\theta + \sin^2\theta} = 1$, thus

$$\int R^2_{\alpha\bar\alpha\alpha\bar\alpha} = 4\int|\alpha_1|^8 d\alpha_1 = 4\int_0^{2\pi} -\sin\theta + i\cos\theta d\theta = 0.$$

On the other hand,

$$\sum|R_{i\bar jk\bar l}|^2 = |R_{1\bar11\bar1}|^2 = 4$$

$$\sum|Ric_{i\bar j}|^2 = \sum|R_{i\bar jk\bar k}|^2 = |R_{1\bar11\bar1}|^2 = 4.$$

Hence

$$0 = 4A + 4B + 4C.$$

Together with identities in (53), we have $A = \frac{4}{15}\pi^2, B = -\frac{32}{15}\pi^2$ and $C = \frac{28}{15}\pi^2$. Hence,

$$\int R^2_{\alpha\bar\alpha\alpha\bar\alpha} = \frac{4}{15}\pi^2\sum_{i,j,k,l}|R_{i\bar jk\bar l}|^2 - \frac{32}{15}\pi^2\sum_{i,j}|Ric_{i\bar j}|^2 + \frac{28}{15}\sum_{i,j,k,l}R_{i\bar ij\bar j}R_{k\bar kl\bar l}. \tag{54}$$

Recall (49) and (50). Together with (54), we have

$$
\begin{aligned}
&\int (R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} - \text{Average} R_{\alpha\bar{\alpha}\alpha\bar{\alpha}})^2 dV(M,g) \\
= &\ \frac{4}{15}\pi^2 \sum_{i,j,k,l} |R_{i\bar{j}k\bar{l}}|^2 - \frac{32}{15}\pi^2 \sum_{i,j} |Ric_{i\bar{j}}|^2 + \frac{28}{15} \sum_{i,j,k,l} R_{i\bar{i}j\bar{j}} R_{k\bar{k}l\bar{l}} \\
&- \frac{16}{9}\pi^2 \sum_{i,j,k,l} R_{i\bar{i}j\bar{j}} R_{k\bar{k}l\bar{l}} + \frac{8}{9}\pi^2 \sum_{i,j,k,l} R_{i\bar{i}j\bar{j}} R_{k\bar{k}l\bar{l}} \\
= &\ \frac{4}{45}\pi^2 \left(3 \sum_{i,j,k,l} |R_{i\bar{j}k\bar{l}}|^2 - 24 \sum_{i,j} |Ric_{i\bar{j}}|^2 + 11 \sum_{i,j} Ric_{i\bar{i}} Ric_{j\bar{j}} \right).
\end{aligned}
$$

Recall that

$$
c_1^2(\Theta) - 3c_2(\Theta) = \frac{1}{8\pi^2}\Big(\sum_{k\neq l, i,j} R_{i\bar{i}k\bar{k}} R_{j\bar{j}l\bar{l}} + 3R_{i\bar{j}k\bar{l}} R_{j\bar{i}l\bar{k}} \Big) dz^k \wedge d\bar{z}^k \wedge dz^l \wedge d\bar{z}^l,
$$

then

$$
\begin{aligned}
&c_1^2(\Theta) - 3c_2(\Theta) \\
= &\ \frac{45}{32\pi^4} \left(\int (R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} - \text{Average} R_{\alpha\bar{\alpha}\alpha\bar{\alpha}})^2 dV \right) dz^k \wedge d\bar{z}^k \wedge dz^l \wedge d\bar{z}^l \\
&+ \left(\frac{3}{\pi^2} \sum_{i,j} |Ric_{i\bar{j}}|^2 - \frac{5}{4\pi^2} \sum_{i,j} Ric_{i\bar{i}} Ric_{j\bar{j}} \right) dz^k \wedge d\bar{z}^k \wedge dz^l \wedge d\bar{z}^l.
\end{aligned}
$$

Integrate over M and obtain,

$$
\begin{aligned}
c_1^2 - 3c_2 &= H \left(\int (R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} - \text{Average} R_{\alpha\bar{\alpha}\alpha\bar{\alpha}})^2 dV \right) K \\
&\quad + \left(\frac{3}{\pi^2} \sum_{i,j} |Ric_{i\bar{j}}|^2 - \frac{5}{4\pi^2} \sum_{i,j} Ric_{i\bar{i}} Ric_{j\bar{j}} \right) K
\end{aligned}
$$

where $H = -\frac{45}{32\pi^4}$ and $K = \int_M dz^k \wedge d\bar{z}^k \wedge dz^l \wedge d\bar{z}^l$. This can be compared with formula (52) where the condition of being Einstein is used.

2. Chern-Number Inequality on Compact Kähler Surfaces of Nonpositive Bisectional Curvatures

This section will give the Chern-number inequality $c_1^2 \geq c_2$ on a compact Kähler surface M of nonpositive bisectional curvature i.e. a compact Kähler manifold whose tangent bundle is nonpositive in the sense of Griffiths. The idea is that we start from the condition of nonpositivity of curvature in the sense of Griffiths of tangent bundle of the M, without knowing the exact expression of the Chern-number inequality. Construct tautologically the associated line bundle L over the projectivized tangent bundle of M, whose curvature contains curvature of TM as a component and is nonpositive in the sense of Griffiths if and only if that of TM is. This is constructed and proved in [Mok] p. 36-39. We will present this construction in the first part of this section. After we get the relation between $TM \longrightarrow M$ with its curvature condition and the $L \longrightarrow \mathbb{P}(TM)$ with its curvature condition. Use the assumption that the curvature of TM is nonpositive in the sense of Griffiths and so is that of L, there will be Chern form inequalities on both bundles caused by the curvature assumption. For L, this implies directly that the first Chern form $c_1(\Theta(L)) \leq 0$. For TM, there will be an inequality in terms of $c_1(\Theta(M))$ and $c_2(\Theta(M))$. To get the specific inequality, we make use of the first Chern class of L, i.e. curvature of L. The main technique involved is fiber integration, which we will see in the second part of this section. Finally, we can get the inequality $c_1^2(\Theta(M)) \geq c_2(\Theta(M))$ for TM and hence the Chern-number inequality $c_1^2 \geq c_2$ for the compact Kähler surface with nonpositive bisectional curvature. The third part of this section will give an example when the equality holds.

2.1. Associated Line Bundle L of TM and the Relation between Curvatures.

2.1.1. *Construction of the Associated Line Bundle.* The following construction with respect to a tangent bundle is a special case of the construction with respect to holomorphic vector bundles dealt in [Mok] p. 38. Suppose M is a compact Kähler surface with metric g, TM be its tangent bundle. Define the *projectivized tangent bundle*

$$\mathbb{P}(TM) \longrightarrow M$$

by projectivizing TM fiber-wisely. That is, the fiber $\mathbb{P}(TM)_x$ over each $x \in M$ is a projectivization of the fiber TM_x as a vector space. Define the associated line bundle $L \longrightarrow \mathbb{P}(TM)$ as a subbundle of the trivial bundle $\mathbb{P}(TM) \times \mathbb{C}^*$ by

$$L = \{((x, [v]), u) : u \in [v], \text{ for all } x \in M, [v] \in \mathbb{P}(TM)_x\}.$$

The associated Hermitian metric $\hat{g}$ of $L \longrightarrow \mathbb{P}(TM)$ can be induced from g as follows: For any fixed $x \in M$, let U be an open neighborhood of x with local coordinates (z_1, z_2). Let $\{e_1, e_2\}$ be the holomorphic frame of TM with coordinates (u_1, u_2). Fix the point $[\mu] = (x, [e_1]) \in \mathbb{P}(TM)$, let V_1 be an open neighborhood of $[\mu]$ in $\mathbb{P}(TM)$ with local homogeneous coordinates $((z_1, z_2), [(u_1, u_2)])$ where $u^1 \neq 0$. Use non-homogeneous coordinates points in V_1 can be written as $((z_1, z_2), v_2)$ where $v_2 = \dfrac{u_2}{u_1}$. Choose the frame f of the line bundle L restricted on V_1 such that the holomorphic coordinate is v_2. Note that such chosen coordinates of $L \longrightarrow \mathbb{P}(TM)$ will automatically be special coordinates at $[\mu]$. Let f^* be the dual of f, then write the induced Hermitian metric $\hat{g}$ on L as

$$\hat{g} = \hat{g}_0 f^* \otimes \bar{f}^*.$$

For any $[\xi] = ((z_1, z_2), v_2) \in V_1$, we may simply define the induced Hermitian metric of L by

$$\hat{g}_{[\xi]}(f, \bar{f}) = g_{(z_1,z_2)}(f, \bar{f}).$$

In particular,

$$\hat{g}_{0[\xi]} = \hat{g}_{0[\xi]} f^* \otimes \bar{f}^*(f, \bar{f}) = \hat{g}_{[\xi]}(f, \bar{f}) = g_{(z_1,z_2)}(f, \bar{f}).$$

2.1.2. *Curvature Relations between L and TM.* Let $\Theta(M)$ denote the curvature of TM of the Hermitian connection with respect to the Kähler metric g of M and let $\Theta(L)$ denote the curvature of L of the Hermitian connection with respect to the induced Hermitian metric $\hat{g}$ of L. Observe that $\Theta(L)$ an $End(L)$-valued $(1,1)$-form over $\mathbb{P}(TM)$, since the basis of $\mathbb{P}(TM)$ with respect to the open set is given by $((z_1, z_2), v_2)$, let dz^1, dz^2, dv^2 denote the dual of the induced frame of $T\mathbb{P}(TM)$, hence $T^*(\mathbb{P}(TM))$ should be with basis

$$\{dz^i \wedge d\bar{z}^j, dz^i \wedge d\bar{v}^2, dv^2 \wedge d\bar{z}^i, dv^2 \wedge d\bar{v}^2\}_{i,j=1,2}.$$

Hence, we may write

$$\Theta(L) = \Theta^{(1)} + \Theta^{(2)} + \Theta^{(3)}$$

where $\Theta^{(1)}$ is the component of $\Theta(L)$ with $dz^i \wedge d\bar{z}^j$ as basis, $\Theta^{(2)}$ is the component of $\Theta(L)$ with $dz^i \wedge d\bar{v}^2$ or $dv^2 \wedge d\bar{z}^i$ as basis and $\Theta^{(3)}$ is component of $\Theta(L)$ with $dv^2 \wedge d\bar{v}^2$ as basis and each of them is with coefficients in $End(L)$. Let us look at $\Theta^{(1)}$ first. It is with basis $\{dz^i \wedge d\bar{z}^j\}$ hence it measures the commutativity of connections in the directions tangent to M as a submanifold of $\mathbb{P}(TM)$. By the constructions of $\mathbb{P}(TM)$, the directions tangent to M as a submanifold of $\mathbb{P}(TM)$ and the directions tangent to M as a manifold itself can be identified, besides, vectors in fibers of L can be identified with vectors in fibers of TM by the identification $((z_1, z_2), v^2) = ((z_1, z_2), (1, v^2)$ also. Hence $\Theta^{(1)}$ will be the same as $\Theta(M)_{11}$. Secondly, let us look at $\Theta^{(2)}$. The mixed terms should commute, i.e. $\dfrac{\partial}{\partial z_i}\dfrac{\partial}{\partial \bar{v}_2} = \dfrac{\partial}{\partial \bar{v}_2}\dfrac{\partial}{\partial z_i}$ and $\dfrac{\partial}{\partial \bar{z}_i}\dfrac{\partial}{\partial v_2} = \dfrac{\partial}{\partial v_2}\dfrac{\partial}{\partial \bar{z}_i}$. Hence curvature in this component should be zero. Finally, let us look at $\Theta^{(3)}$, since its basis is $dv^2 \wedge d\bar{v}^2$, it should measure the commutativity of connections in the directions normal to M as a submanifold of $\mathbb{P}(TM)$, these are exactly the directions along the fibers of the $\mathbb{P}(TM)$ when considered as a vector bundle over M. Hence $\Theta^{(3)}$ evaluate at each fiber of $\mathbb{P}(TM)$ will equal to the curvature of the tautological line bundle of $\mathbb{P}^1_{\mathbb{C}}$ with Fubini-Study metric and which is fiber-wisely constant. Hence $\Theta^{(3)} = -1 dv^2 \wedge d\bar{v}^2$. From the above discussion we have:

PROPOSITION 2.1. *Evaluate at $[\mu]$,*

$$\Theta(L)([\mu]) = -dv^2 \wedge d\bar{v}^2 + \sum_{i,j} \Theta(M)_1{}^1{}_{i\bar{j}} dz^i \wedge d\bar{z}^j. \tag{55}$$

Alternatively, this proposition can be proven by representing the curvatures by the metrics and using the relation between metrics to get the identity of curvatures:

PROOF. Under the frame f of L, let $\Theta(L)_f = (\Theta_1{}^1)$ denote the curvature of the Hermitian connection of L and let $H_f = (\hat{g}_0)$ denote the metric matrix, hence from Proposition 2.1 in Chapter 2, we have

$$\Theta(L)_f = -\partial\bar{\partial} H_f \cdot H_f^{-1} + \partial H_f \wedge \bar{\partial} H_f \cdot H_f^{-2}.$$

Since now each matrix is of only one element, we have

$$\Theta_1{}^1 = -\partial\bar{\partial}\hat{g}_0 \cdot \hat{g}_0^{-1} + \partial\hat{g}_0 \wedge \bar{\partial}\hat{g}_0 \cdot \hat{g}_0^{-2}.$$

Since the coordinate at $[\mu]$ is special coordinate hence $dg_0([\mu]) = 0$ and hence $\partial g_0([\mu]) = 0$ and $\bar{\partial} g_0([\mu]) = 0$, hence

$$\Theta_1{}^1[\mu] = -\partial\bar{\partial}\hat{g}_0([\mu]) \cdot \hat{g}_0([\mu])^{-1}.$$

Since at any $[\xi] = ((z_1, z_2), [v_2]) \in V_1$,

$$\begin{aligned}
\hat{g}_{0[\xi]} &= g_{(z_1,z_2)}(f, \bar{f}) \\
&= g_{(z_1,z_2)}((1, v_2), \overline{(1, v_2)}) \\
&= g_{(z_1,z_2)1\bar{1}} + g_{(z_1,z_2)1\bar{2}}\bar{v}_2 + g_{(z_1,z_2)2\bar{1}}v_2 + g_{(z_1,z_2)2\bar{2}}v_2\bar{v}_2
\end{aligned}$$

Evaluate at $[\mu]$, $\hat{g}_{0[\mu]} = 1$. We may omit the subscript $_{(z_1,z_2)}$ of $g_{(z_1,z_2)}$ in the following computation.

$$\begin{aligned}
\bar{\partial}\hat{g}_{0[\xi]} &= \sum_{j=1}^{2} \frac{\partial}{\partial \bar{z}_j}(g_{1\bar{1}} + g_{1\bar{2}}\bar{v}_2 + g_{2\bar{1}}v_2 + g_{2\bar{2}}v_2\bar{v}_2)d\bar{z}^j \\
&\quad + \frac{\partial}{\partial \bar{v}_2}(g_{1\bar{1}} + g_{1\bar{2}}\bar{v}_2 + g_{2\bar{1}}v_2 + g_{2\bar{2}}v_2\bar{v}_2)d\bar{v}^2 \\
&= \sum_{j=1}^{2} \frac{\partial}{\partial \bar{z}_j} g_{1\bar{1}} d\bar{z}^j + \bar{v}_2 \sum_{i=j}^{2} \frac{\partial}{\partial \bar{z}_j} g_{1\bar{2}} d\bar{z}^j \\
&\quad + v_2 \sum_{j=1}^{2} \frac{\partial}{\partial \bar{z}_j} g_{2\bar{1}} d\bar{z}^j + v_2\bar{v}_2 \sum_{j=1}^{2} \frac{\partial}{\partial \bar{z}_j} g_{2\bar{2}} d\bar{z}^j \\
&\quad + g_{1\bar{2}} d\bar{v}^2 + g_{2\bar{2}} v_2 d\bar{v}^2
\end{aligned}$$

$$\begin{aligned}
\partial\bar{\partial} g_{0[\xi]} &= \sum_{i,j=1}^{2} \frac{\partial^2 g_{1\bar{1}}}{\partial z_i \partial \bar{z}_j} dz^i \wedge d\bar{z}^j + \sum_{j=1}^{2} \frac{\partial^2 g_{1\bar{1}}}{\partial v_2 \partial \bar{z}_j} dv^2 \wedge d\bar{z}^j \\
&\quad + \bar{v}_2 \sum_{i,j=1}^{2} \frac{\partial^2 g_{1\bar{2}}}{\partial z_i \partial \bar{z}_j} dz^i \wedge d\bar{z}^j + \sum_{i=1}^{2} \frac{\partial}{\partial v_2} \bar{v}_2 \frac{\partial}{\partial \bar{z}_j} g_{1\bar{2}} dv^2 \wedge d\bar{z}^j \\
&\quad + v_2 \sum_{i,j=1}^{2} \frac{\partial^2 g_{2\bar{1}}}{\partial z_i \partial \bar{z}_j} dz^i \wedge d\bar{z}^j + \frac{\partial}{\partial v_2} v_2 \sum_{j=1}^{2} \frac{\partial}{\partial \bar{z}_j} g_{2\bar{1}} dv^2 \wedge d\bar{z}^j \\
&\quad + v_2\bar{v}_2 \sum_{i,j=1}^{2} \frac{\partial^2 g_{2\bar{2}}}{\partial z_i \partial \bar{z}_j} dz^i \wedge d\bar{z}^j + \frac{\partial}{\partial v_2} v_2\bar{v}_2 \sum_{j=1}^{2} \frac{\partial}{\partial \bar{z}_j} g_{2\bar{2}} dv^2 \wedge d\bar{z}^j \\
&\quad + \sum_{i=1}^{2} \frac{\partial g_{1\bar{2}}}{\partial z_i} dz^i \wedge d\bar{v}^2 + \frac{\partial g_{1\bar{2}}}{\partial v_2} dv^2 \wedge d\bar{v}^2 \\
&\quad + v_2 \sum_{i=1}^{2} \frac{\partial g_{2\bar{2}}}{\partial z_i} dz^i \wedge d\bar{v}^2 + \frac{\partial}{\partial v_2} g_{2\bar{2}} v_2 dv^2 \wedge d\bar{v}^2
\end{aligned}$$

Evaluate at $[\mu]$,

$$\partial\bar{\partial} g_{0[\mu]} = \sum_{i,j=1}^{2} \frac{\partial^2 g_{1\bar{1}}}{\partial z_i \partial \bar{z}_j} dz^i \wedge d\bar{z}^j + dv^2 \wedge d\bar{v}^2$$

Hence

$$\begin{aligned}
{\Theta_1}^1[\mu] &= -\partial\bar{\partial}\hat{g}_0([\mu])\cdot\hat{g}_0([\mu])^{-1} \\
&= -\sum_{i,j=1}^{2}\frac{\partial^2 g_{1\bar{1}}}{\partial z_i\partial\bar{z}_j}dz^i\wedge d\bar{z}^j - dv^2\wedge d\bar{v}^2 \\
&= \sum_{i,j=1}^{2}\Theta(M)_1{}^1{}_{i\bar{j}}dz^i\wedge d\bar{z}^j - dv^2\wedge d\bar{v}^2
\end{aligned}$$

□

2.2. Get Chern-Form Inequality of TM from that of L. From (55), we have that at $[\mu]$, curvature of TM is nonpositive in the sense of Griffiths if and only if curvature of L is nonpositive in the sense of Griffiths. If M is a compact Kähler surface with nonpositive bisectional curvature at (z_1,z_2), then $\Theta(L)([\mu])$ is nonpositive in the sense of Griffiths. Hence the first Chern form $c_1(\Theta(L))=\frac{\sqrt{-1}}{2\pi}\Theta(L)([\mu])$ of L is nonpositive. To get the corresponding Chern form inequality on M, we will make use of the formula (55): write $v^2=w$, we have ${\Theta_1}^1[\mu]=-dw\wedge d\bar{w}+\sum_{i,j=1}^{2}{\Theta_1}^1{}_{i\bar{j}}dz^i\wedge d\bar{z}^j$. Hence,

$$\wedge^3\Theta(L)[\mu] = -3dw\wedge d\bar{w}\left(\sum_{1\le i,j\le 2}{\Theta_r}^r{}_{i\bar{j}}dz^i\wedge d\bar{z}^j\right)^2,$$

which is a $(3,3)$-form on $\mathbb{P}(TM)$. Note that it is greater than or equal to zero by the curvature assumption. To get the corresponding curvature form inequality at the point (z_1,z_2), we have to (i) get rid of the $dw\wedge d\bar{w}$ by taking fiber integration of $\wedge^3\Theta(L)[\mu]$ along the fiber of $L_{[\mu]}$ and (ii) taking average in all directions in $\mathbb{P}(TM)_x$ by taking integration with respect to $[\mu]$ which we will write r instead of 1 in the following context. Hence,

$$\begin{aligned}
&\int_{\mathbb{P}(T_xX)}\int_{L_{[r]}}\wedge^3{\Theta_1}^1[r] \\
=\;& \int_{\mathbb{P}(T_xX)}\int_{\mathbb{P}^1}-3dw\wedge d\bar{w}\left(\sum_{1\le i,j\le 2}{\Theta_r}^r{}_{i\bar{j}}dz^i\wedge d\bar{z}^j\right)^2 \\
=\;& (-3)\int_{\mathbb{P}(TX)}\left(\sum_{1\le i,j\le 2}\Theta^r_{ri\bar{j}}dz^i\wedge d\bar{z}^j\right)^2\underbrace{\int_{\mathbb{P}^1}dw\wedge d\bar{w}}_{C_1} \\
=\;& (-3)\cdot C_1\cdot\int_{\mathbb{P}(T_xX)}\left(\sum_{1\le i,j\le 2}{\Theta_r}^r{}_{i\bar{j}}dz^i\wedge d\bar{z}^j\right)^2 \\
=\;& (-3)\cdot C_1\cdot\int_{\mathbb{P}(T_xX)}({\Theta_r}^r{}_{i\bar{i}}\Theta^r_{rj\bar{j}}-{\Theta_r}^r{}_{i\bar{j}}{\Theta_r}^r{}_{j\bar{i}})dz^i\wedge d\bar{z}^i\wedge dz^j\wedge d\bar{z}^j \\
=\;& (-3)\cdot C_1\cdot\sum_{1\le k,l\le 2}\int_{\mathbb{P}(T_xX)}(-{\Theta_r}^r{}_{k\bar{l}}{\Theta_r}^r{}_{l\bar{k}}+{\Theta_r}^r{}_{k\bar{k}}{\Theta_r}^r{}_{l\bar{l}})dz^k\wedge d\bar{z}^k\wedge dz^l\wedge d\bar{z}^l
\end{aligned}$$

Represent by curvature tensor: ${\Theta_r}^r{}_{k\bar{l}} = R_{i\bar{j}k\bar{l}}\alpha^i\bar{\alpha}^j, 1 \le i, j \le 2$, then

$$\begin{aligned}
-{\Theta_r}^r{}_{k\bar{l}}{\Theta_r}^r{}_{l\bar{k}} &= -R_{i\bar{j}k\bar{l}}R_{r\bar{s}l\bar{k}}\alpha^i\bar{\alpha}^j\alpha^r\bar{\alpha}^s \\
&= -(R_{1\bar{2}k\bar{l}}R_{2\bar{1}l\bar{k}} + R_{2\bar{1}k\bar{l}}R_{1\bar{2}l\bar{k}} + R_{1\bar{1}k\bar{l}}R_{2\bar{2}l\bar{k}} + R_{2\bar{2}k\bar{l}}R_{1\bar{1}l\bar{k}})|\alpha^1|^2|\alpha^2|^4, \\
{\Theta_r}^r{}_{k\bar{k}}{\Theta_r}^r{}_{l\bar{l}} &= R_{i\bar{j}k\bar{k}}R_{r\bar{s}l\bar{l}}\alpha^i\bar{\alpha}^j\alpha^r\bar{\alpha}^s \\
&= (R_{1\bar{2}k\bar{k}}R_{2\bar{1}l\bar{l}} + R_{2\bar{1}k\bar{k}}R_{1\bar{2}l\bar{l}} + R_{1\bar{1}k\bar{k}}R_{2\bar{2}l\bar{l}} + R_{2\bar{2}k\bar{k}}R_{1\bar{1}l\bar{l}})|\alpha^1|^2|\alpha^2|^2.
\end{aligned}$$

Since $\mathbb{P}T_xX \cong S^3$, then let $C_2 = \int_{S^3} |\alpha^1|^2|\alpha^2|^2$. Hence we have

$$\begin{aligned}
&\int_{\mathbb{P}(T_xX)}\int_{L_{[\mu]}} \wedge^3\Theta(L)([r]) \\
= \;& (-3)\underbrace{C_1}_{>0}\underbrace{C_2}_{>0}\sum_{k,l}(-R_{1\bar{2}k\bar{l}}R_{2\bar{1}l\bar{k}} - R_{2\bar{1}k\bar{l}}R_{1\bar{2}l\bar{k}} - R_{1\bar{1}k\bar{l}}R_{2\bar{2}l\bar{k}} - R_{2\bar{2}k\bar{l}}R_{1\bar{1}l\bar{k}} \\
&+R_{1\bar{2}k\bar{k}}R_{2\bar{1}l\bar{l}} + R_{2\bar{1}k\bar{k}}R_{1\bar{2}l\bar{l}} + R_{1\bar{1}k\bar{k}}R_{2\bar{2}l\bar{l}} + R_{2\bar{2}k\bar{k}}R_{1\bar{1}l\bar{l}})dz^k \wedge d\bar{z}^k \wedge dz^l \wedge d\bar{z}^l \\
= \;& (-3)\underbrace{C_1}_{>0}\underbrace{C_2}_{>0}\sum_{i,j,k,l}(-R_{i\bar{j}k\bar{l}}R_{j\bar{i}l\bar{k}} - R_{i\bar{i}k\bar{l}}R_{j\bar{j}l\bar{k}} + R_{i\bar{j}k\bar{k}}R_{j\bar{i}l\bar{l}} + R_{i\bar{i}k\bar{k}}R_{j\bar{j}l\bar{l}}) \\
& dz^k \wedge d\bar{z}^k \wedge dz^l \wedge d\bar{z}^l \\
= \;& (-3)\underbrace{C_1}_{>0}\underbrace{C_2}_{>0}\sum_{i,j}(Ric_{i\bar{i}}Ric_{j\bar{j}} + Ric_{j\bar{i}}Ric_{i\bar{j}})
\end{aligned}$$

On the other hand,

$$c_1(\Theta(M)) = \sum_i \frac{\sqrt{-1}}{2\pi}Ric_{i\bar{i}},$$

$$c_2(\Theta(M)) = \frac{-1}{8\pi^2}\sum_{i,j}(Ric_{i\bar{i}} \wedge Ric_{j\bar{j}} - Ric_{i\bar{j}} \wedge Ric_{j\bar{i}}),$$

gives

$$(c_1^2(\Theta(M)) - c_2(\Theta(M))) = \frac{-1}{8\pi^2}\sum_{i,j}(Ric_{i\bar{i}} \wedge Ric_{j\bar{j}} + Ric_{i\bar{j}} \wedge Ric_{j\bar{i}})$$

Therefore,

$$-8\pi^2(c_1^2((\Theta(M)) - c_2(\Theta(M))_x = \frac{1}{-3 \cdot C_1C_2}\int_{\mathbb{P}T_xX}\int_{S^3} \wedge^3{\Theta_1}^1[r].$$

Since $\wedge^3{\Theta_1}^1[r] \ge 0$, then $c_1^2((\Theta(M)) \ge c_2((\Theta(M))$ and hence we get the Chern number inequality $c_1^2 \ge c_2$ for a compact Kähler surface M of nonposistive bisectional curvature.

2.3. Case when Equality Holds. In the case that the equality holds, that is $c_1^2 = c_2$, we would give an example of a compact Kähler surface of non-positive bisectional curvature where $c_1^2 = c_2$. When bisectional curvatures are nonnegative/nonpositive and $c_1^2 = c_2$ if and only if $c_1(\Theta(L)) = 0$ by calculation in the last subsection.

Set up: Let Γ be a lattice such that $\mathbb{C}^3/\Gamma$ is a compact complex torus. Let M be a submanifold of dimension 2 of $\mathbb{C}^3/\Gamma$. We will show $c_1^2 = c_2$ on $(M, g|_M)$

where g is the metric of $\mathbb{C}^3/\Gamma$ by showing $c_1^3 = 0$ on the associated line bundle $(L, \hat{g})$ of TM.

Proof. By the formula,

$$\Theta(L,\hat{g})[\mu] = -dw \wedge d\bar{w} + \sum_{1\le i,j\le 2} {\Theta_1}^1{}_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

where $[\mu] = (x, [v^2]) \in \mathbb{P}(TM)$ with non-homogeneous coordinates $((z_1, z_2), v^2)$. Then,

$$\begin{aligned}
&\frac{8\pi^2}{\sqrt{-1}}(-c_1(\Theta(L)))^3 \\
= \;& (\Theta(L,\hat{h})[\mu])^3 \\
= \;& -3dw \wedge d\bar{w} \wedge \left(\sum_{1\le i,j\le 2} {\Theta_1}^1{}_{i\bar{j}} dz^i \wedge d\bar{z}^j\right)^2 \\
= \;& -3dw \wedge d\bar{w} \wedge \left(\sum_{1\le i,j\le 2} {\Theta_1}^1{}_{i\bar{j}} dz^i \wedge d\bar{z}^j\right) \wedge \left(\sum_{1\le k,l\le 2} {\Theta_1}^1{}_{k\bar{l}} dz^k \wedge d\bar{z}^l\right) \\
= \;& -3dw \wedge d\bar{w} \wedge \left(\sum_{1\le i,j\le 2, i\ne j} {\Theta_1}^1{}_{i\bar{i}}{\Theta_1}^1{}_{j\bar{j}} - {\Theta_1}^1{}_{i\bar{j}}{\Theta_1}^1{}_{j\bar{i}}\right) dz^i \wedge d\bar{z}^i \wedge dz^j \wedge d\bar{z}^j \\
= \;& -6dw \wedge d\bar{w} \wedge ({\Theta_1}^1{}_{1\bar{1}}{\Theta_1}^1{}_{2\bar{2}} - {\Theta_1}^1{}_{1\bar{2}}{\Theta_1}^1{}_{2\bar{1}}) dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2.
\end{aligned}$$

Represent by matrix

$$-\frac{8\pi^2}{\sqrt{-1}}(-c_1)^3 = det \begin{pmatrix} dw \wedge d\bar{w} & * & * \\ * & {\Theta_1}^1{}_{1\bar{1}} dz^1 \wedge d\bar{z}^1 & {\Theta_1}^1{}_{1\bar{2}} dz^1 \wedge d\bar{z}^2 \\ * & {\Theta_1}^1{}_{2\bar{1}} dz^2 \wedge d\bar{z}^1 & {\Theta_1}^1{}_{2\bar{2}} dz^2 \wedge d\bar{z}^2 \end{pmatrix}$$

But the minor

$$\begin{pmatrix} {\Theta_1}^1{}_{1\bar{1}} dz^1 \wedge d\bar{z}^1 & {\Theta_1}^1{}_{1\bar{2}} dz^1 \wedge d\bar{z}^2 \\ {\Theta_1}^1{}_{2\bar{1}} dz^2 \wedge d\bar{z}^1 & {\Theta_1}^1{}_{2\bar{2}} dz^2 \wedge d\bar{z}^2 \end{pmatrix}$$

has rank 1, so there exist zero eigenvalue and hence $c_1^3(\Theta(L)) = 0$ and hence $c_1^2 = c_2$ for M. □

3. Chern-Number Inequality on Compact Kähler Surfaces of Nonpositive Riemannian Sectional Curvatures

We will show that when a compact Kähler surface is of nonpositive Riemannian sectional curvature, then $c_1^2 \ge 2c_2$ on M. We will use the equivalence of positivity of the Riemannian sectional curvature and the *curvature operator* (see Theorem 9.26 in [Zheng] p. 242) for complex two dimension manifolds and prove the inequality through the nonpositivity of the curvature operator. Secondly, we will show that when the equality holds, the nonpositive Riemannian sectional curvatured compact Kähler surface is biholomorphic to $\Delta \times \Delta$.

3.1. $c_1^2 \geq 2c_2$ on a Compact Kähler Surfaces of Nonpositive Riemannian Sectional Curvature.

DEFINITION 3.1 (Curvature Operator). Let M be a Kähler surface and R its complexified Riemannian curvature tensor. A *curvature operator* Q of M is a Hermitian bilinear form

$$Q : (T^{\mathbb{R}}M \otimes \overline{T^{\mathbb{R}}M}) \times (T^{\mathbb{R}}M \otimes \overline{T^{\mathbb{R}}M}) \longrightarrow \mathbb{C},$$

defined by $Q(u \otimes \bar{v}, w \otimes \bar{z}) = R(u, \bar{v}, w, \bar{z})$ for $u \otimes \bar{v}, w \otimes \bar{z} \in T^{\mathbb{R}}M \otimes \overline{T^{\mathbb{R}}M}$.

Let M be a compact Kähler surface of nonpositive Riemannian sectional curvature, then the corresponding curvature operator Q of M will be non-negative. Let $\{e_1, e_2\}$ be a holomorphic frame of $T^{\mathbb{R}}M$, under the basis $\{e_1 \otimes \bar{e}_1, e_2 \otimes \bar{e}_2, e_1 \otimes \bar{e}_2, e_2 \otimes \bar{e}_1\}$ of $T^{\mathbb{R}}M \otimes \overline{T^{\mathbb{R}}M}$, Q has the matrix representation with entries the complexified Riemannian curvature tensors,

$$Q = \begin{pmatrix} a & b & \bar{e} & e \\ b & c & \bar{f} & f \\ e & f & b & d \\ \bar{e} & \bar{f} & \bar{d} & b \end{pmatrix} = \begin{array}{c} \\ 11 \\ 22 \\ 12 \\ 21 \end{array} \begin{array}{c} \begin{array}{cccc} 11 & 22 & 21 & 12 \end{array} \\ \begin{pmatrix} R_{1\bar{1}1\bar{1}} & R_{1\bar{1}2\bar{2}} & R_{1\bar{1}2\bar{1}} & R_{1\bar{1}1\bar{2}} \\ R_{2\bar{2}1\bar{1}} & R_{2\bar{2}2\bar{2}} & R_{2\bar{2}2\bar{1}} & R_{2\bar{2}1\bar{2}} \\ R_{1\bar{2}1\bar{1}} & R_{1\bar{2}2\bar{2}} & R_{1\bar{2}2\bar{1}} & R_{1\bar{2}1\bar{2}} \\ R_{2\bar{1}1\bar{1}} & R_{2\bar{1}2\bar{2}} & R_{2\bar{1}2\bar{1}} & R_{2\bar{1}1\bar{2}} \end{pmatrix} \end{array}.$$

A vector $u \otimes \bar{v} \in T^{\mathbb{R}}M \otimes \overline{T^{\mathbb{R}}M}$ is said to be *of trace-free* if and only if $v^*(u) = 0$. Let $\tilde{T}$ denote the subspace of $T^{\mathbb{R}}M \otimes \overline{T^{\mathbb{R}}M}$ consisting of all trace free elements. Thus $\tilde{T}$ will be of dimension 3. Note that $\tilde{T}$ will correspond to the trace free part $End_0(TM)$ of $End(TM)$. We call the complement of $End_0(TM)$ in $End(TM)$ to be the *trivial part*. Curvature tensor as an $End(TM)$-valued $(1,1)$-form on M, the contribution of the trivial part is scaling the curvature by the length of vectors we measured. Hence all the essential properties of curvature lies in the trace-free part. To get topological invariants related to curvatures, it suffices to consider the space $\tilde{T}$. Choose the following basis for $\tilde{T}$,

$$\begin{cases} u_1 = & e_1 \otimes \bar{e}_2 \\ u_2 = & e_2 \otimes \bar{e}_1 \\ u_3 = & \frac{1}{\sqrt{2}} e_1 \otimes \bar{e}_1 - \frac{1}{\sqrt{2}} e_2 \otimes \bar{e}_2 \end{cases}$$

Denote the restriction of the curvature operator Q on the subspace $\tilde{T}$ by

$$\tilde{Q} = Q|_{\tilde{T}} : \tilde{T} \longrightarrow \mathbb{C}.$$

Under the basis $\{u_1, u_2, u_3\}$ chosen for $\tilde{T}$, $\tilde{Q}$ can be represented in a matrix form $\tilde{Q} = (\tilde{Q}_{ij})_{3\times 3}$ with $\tilde{Q}_{ij} = Q(u_i, u_j), i, j = 1, 2, 3$. Compute,

$$\tilde{Q}_{11} = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & b & \bar{e} & e \\ b & c & \bar{f} & f \\ e & f & b & d \\ \bar{e} & \bar{f} & \bar{d} & b \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = b,$$

$$\tilde{Q}_{12} = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & b & \bar{e} & e \\ b & c & \bar{f} & f \\ e & f & b & d \\ \bar{e} & \bar{f} & \bar{d} & b \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = d,$$

$$\tilde{Q}_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & b & \bar{e} & e \\ b & c & \bar{f} & f \\ e & f & b & d \\ \bar{e} & \bar{f} & \bar{d} & b \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(e-f).$$

Computation for other entries is similar. Hence we obtain,

$$\tilde{Q} = \begin{pmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} & \tilde{Q}_{13} \\ \tilde{Q}_{21} & \tilde{Q}_{22} & \tilde{Q}_{23} \\ \tilde{Q}_{31} & \tilde{Q}_{32} & \tilde{Q}_{33} \end{pmatrix} = \begin{pmatrix} b & d & \frac{1}{\sqrt{2}}(e-f) \\ \bar{d} & b & \frac{1}{\sqrt{2}}(\bar{e}-\bar{f}) \\ \frac{1}{\sqrt{2}}(\bar{e}-\bar{f}) & \frac{1}{\sqrt{2}}(e-f) & \frac{1}{2}(a+c-2b) \end{pmatrix}. \tag{56}$$

Since $Q \leq 0$, then its restriction $\tilde{Q} \leq 0$. Therefore, $\sum_{k=1}^{3} det((2 \times 2)\text{-minor } M_k \text{ of } \tilde{Q}) \leq 0$. Thus,

$$\begin{aligned} 0 \;\geq\; & \underbrace{(\tilde{Q}_{11}\tilde{Q}_{22} - \tilde{Q}_{12}\tilde{Q}_{21})}_{detM_1} + \underbrace{(\tilde{Q}_{22}\tilde{Q}_{33} - \tilde{Q}_{23}\tilde{Q}_{32})}_{detM_3} + \underbrace{(\tilde{Q}_{11}\tilde{Q}_{33} - \tilde{Q}_{13}\tilde{Q}_{31})}_{detM_2} \\ = \;& (b^2 - d\bar{d}) + (\frac{1}{2}ab + \frac{1}{2}bc - b^2 - \frac{1}{2}(e-f)(\bar{e}-\bar{f})) \\ & + (\frac{1}{2}ab + \frac{1}{2}bc - b^2 - \frac{1}{2}(e-f)(\bar{e}-\bar{f})) \\ = \;& (b^2 - d\bar{d}) + (ab + bc - 2b^2 - e\bar{e} + e\bar{f} + \bar{e}f - f\bar{f}) \\ = \;& (R_{1\bar{1}2\bar{2}}^2 - R_{1\bar{2}1\bar{2}}R_{2\bar{1}2\bar{1}}) + R_{1\bar{1}1\bar{1}}R_{1\bar{1}2\bar{2}} + R_{1\bar{1}2\bar{2}}R_{2\bar{2}2\bar{2}} - 2R_{1\bar{1}2\bar{2}}^2 \\ & - R_{1\bar{2}1\bar{1}}R_{2\bar{1}1\bar{1}} + R_{1\bar{2}1\bar{1}}R_{2\bar{1}2\bar{2}} + R_{2\bar{1}1\bar{1}}R_{1\bar{2}2\bar{2}} - R_{1\bar{2}2\bar{2}}R_{2\bar{2}2\bar{1}} \\ = \;& R_{1\bar{1}1\bar{1}}R_{1\bar{1}2\bar{2}} - R_{1\bar{1}1\bar{2}}R_{1\bar{1}2\bar{1}} + R_{2\bar{2}1\bar{1}}R_{2\bar{2}2\bar{2}} - R_{2\bar{2}1\bar{2}}R_{2\bar{2}2\bar{1}} \\ & + R_{1\bar{2}1\bar{1}}R_{2\bar{1}2\bar{2}} - R_{1\bar{2}1\bar{2}}R_{2\bar{1}2\bar{1}} + R_{1\bar{2}2\bar{2}}R_{2\bar{1}1\bar{1}} - R_{1\bar{2}2\bar{1}}R_{2\bar{1}1\bar{2}} \end{aligned}$$

On the other hand,

$$c_1(\Theta) = \frac{\sqrt{-1}}{2\pi}(Ric_{1\bar{1}} + Ric_{2\bar{2}}),$$

$$c_2(\Theta) = \frac{-1}{4\pi^2}(Ric_{1\bar{1}} \wedge Ric_{2\bar{2}} - Ric_{1\bar{2}} \wedge Ric_{2\bar{1}})$$

gives

$$\begin{aligned} c_1(\Theta)^2 - 2c_2(\Theta) \;=\;& \frac{-1}{4\pi^2}(\wedge^2 Ric_{1\bar{1}} + \wedge^2 Ric_{2\bar{2}} + 2Ric_{1\bar{2}} \wedge Ric_{2\bar{1}}) \\ =\;& \frac{-1}{2\pi^2}(R_{1\bar{1}1\bar{1}}R_{1\bar{1}2\bar{2}} - R_{1\bar{1}1\bar{2}}R_{1\bar{1}2\bar{1}} + R_{2\bar{2}1\bar{1}}R_{2\bar{2}2\bar{2}} - R_{2\bar{2}1\bar{2}}R_{2\bar{2}2\bar{1}} \\ & + R_{1\bar{2}1\bar{1}}R_{2\bar{1}2\bar{2}} - R_{1\bar{2}1\bar{2}}R_{2\bar{1}2\bar{1}} + R_{1\bar{2}2\bar{2}}R_{2\bar{1}1\bar{1}} - R_{1\bar{2}2\bar{1}}R_{2\bar{1}1\bar{2}}) \\ & dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2. \end{aligned}$$

Take integration on both sides over M, we get the Chern numbers of M given by

$$\begin{aligned} c_1^2 - 2c_2 &= \frac{-1}{2\pi^2}(R_{1\bar{1}1\bar{1}}R_{1\bar{1}2\bar{2}} - R_{1\bar{1}1\bar{2}}R_{1\bar{1}2\bar{1}} + R_{2\bar{2}1\bar{1}}R_{2\bar{2}2\bar{2}} - R_{2\bar{2}1\bar{2}}R_{2\bar{2}2\bar{1}} \\ &\quad + R_{1\bar{2}1\bar{1}}R_{2\bar{1}2\bar{2}} - R_{1\bar{2}1\bar{2}}R_{2\bar{1}2\bar{1}} + R_{1\bar{2}2\bar{2}}R_{2\bar{1}1\bar{1}} - R_{1\bar{2}2\bar{1}}R_{2\bar{1}1\bar{2}}) \\ &\quad \int_M dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 \geq 0. \end{aligned}$$

That is $c_1^2 \geq 2c_2$ for a compact Kähler surface with nonpositive Riemannian sectional curvatures.

3.2. Case when Equality Holds. In the case that the equality holds, that is $c_1^2 = 2c_2$. We would show that M is biholomorphic to the $\Delta \times \Delta$ where Δ stands for the unit disk under the condition of nonpositive Riemannian sectional curvature. To realize this, we will first use the curvature conditions given by $c_1(\Theta)^2 = 2c_2(\Theta)$ and nonpositivity of Riemannian sectional curvature to find a proper frame for TM such that the frame can split TM into two components with both of them being parallel. Then by the *de Rham decomposition theorem*, which we can only state without proof, we can get the product structure of M from the two components in TM. The same reason as before, it suffices to work on the space $\tilde{T}$ instead of TM itself.

3.2.1. *Properties from $c_1^2 = 2c_2$.* Consider the matrix (56) of $\tilde{Q}$, $c_1^2 = 2c_2$ implies $c_1(\Theta)^2 = 2c_2(\Theta)$. This implies that the determinants of the 2×2-minors M_1, M_2, M_3 are all zero. This implies that at least two of the eigenvalues equal 0. Thus b must be zero. After changing of coordinates if necessary, let $e = \{e_1, e_2\}$ be a frame of $T^{\mathbb{R}}M$ such that the induced basis $\{u_1, u_2, u_3\}$ can diagonalize the matrix $\tilde{Q}$. Hence under the frame e, we have essentially $b = 0, d = 0, e = f$, which gives up to symmetric transformation of curvature tensor,

$$R^e_{1\bar{1}2\bar{2}} = 0, \quad R^e_{1\bar{2}1\bar{2}} = 0, \quad R^e_{1\bar{2}1\bar{1}} = R^e_{1\bar{2}2\bar{2}}. \tag{57}$$

3.2.2. *Properties from Nonpositivity of Riemannian Sectional Curvature of M.* Consider the space of holomorphic frames $G = \{f : f = \{f_1, f_2\}\}$ and a function

$$\varphi : G \times \tilde{T} \longrightarrow \mathbb{C}$$

defined by $\varphi(f, (u \otimes \bar{v})) = R^f_{u\bar{u}v\bar{v}}$ where R^f stands for the curvature tensor with respect to the holomorphic frame f and $u \otimes \bar{v} \in \tilde{T}$. By the assumption that M is of nonpositive Riemannian sectional curvature, then $\varphi(f, (u\otimes\bar{v})) \leq 0$ for any holomorphic frame $f = \{f_1, f_2\}$ and any $u \otimes \bar{v}$. Hence the frame $e = \{e_1, e_2\}$ which gives (57) found in the last subsubsection with vector $e_1 \otimes \bar{e}_2 \in \tilde{T}$ will be a critical point in $G \times \tilde{T}$ of the function φ. Under this frame $e = \{e_1, e_2\}$ we have that all $R^e_{i\bar{j}k\bar{l}}$ involving three 1's, one 2 and three 2's, one 1 will vanish by the vanishing of gradient of φ at $(e, e_1 \otimes \bar{e}_2) \in G \times \tilde{T}$. Indeed, $dR^e_{1\bar{1}2\bar{2}} = 0$

implies for any $\varepsilon \in \mathbb{C}$ with sufficiently small norm,

$$\begin{aligned} 0 &= R^e(e_1 + \varepsilon e_2, \bar{e}_1 + \bar{\varepsilon}\bar{e}_2, e_2, \bar{e}_2) \\ &= R^e_{1\bar{1}2\bar{2}} + \bar{\varepsilon} R^e_{1\bar{2}2\bar{2}} + \varepsilon R^e_{2\bar{1}2\bar{2}} + \varepsilon\bar{\varepsilon} R^e_{2\bar{2}2\bar{2}} \end{aligned}$$

First order approximation gives $R^e_{1\bar{2}2\bar{2}} = 0$. Other terms can be checked similarly. Hence the only possible non-vanishing curvature tensor for $\{e_1, e_2\}$ will be $R^e_{1\bar{1}1\bar{1}}$ and $R^e_{2\bar{2}2\bar{2}}$. Further since e is critical point for φ, hence $R^e_{u\bar{u}v\bar{v}}$ will be constant. Hence $R^e_{1\bar{1}1\bar{1}}$ and $R^e_{2\bar{2}2\bar{2}}$ are both constants.

3.2.3. *Parallel Components of TM.* With the frame $e = \{e_1, e_2\}$ found above, for any $x \in M$, define T' to be the subbundle of $T^{\mathbb{R}}(M)$ with frame e_1 and T'' to be the subbundle of $T^{\mathbb{R}}(M)$ with frame e_2. We will show that the two subbundles T' and T'' are both parallel. It suffices to check the cases of basis, that is,

$$A = \begin{cases} \text{(a) } \nabla_1 e_1 \perp e_2 \\ \text{(b) } \nabla_{\bar{1}} e_1 \perp e_2 \\ \text{(c) } \nabla_2 e_1 \perp e_2 \\ \text{(d) } \nabla_{\bar{2}} e_1 \perp e_2 \end{cases} \quad B = \begin{cases} \text{(a}'\text{) } \nabla_1 e_2 \perp e_1 \\ \text{(b}'\text{) } \nabla_{\bar{1}} e_2 \perp e_1 \\ \text{(c}'\text{) } \nabla_2 e_2 \perp e_1 \\ \text{(d}'\text{) } \nabla_{\bar{2}} e_2 \perp e_1 \end{cases}$$

$$C = \begin{cases} \nabla_{\bar{1}} e_{\bar{1}} \perp \bar{e}_2 \\ \nabla_1 \bar{e}_1 \perp \bar{e}_2 \\ \nabla_{\bar{2}} \bar{e}_1 \perp \bar{e}_2 \\ \nabla_2 \bar{e}_1 \perp \bar{e}_2 \end{cases} \quad D = \begin{cases} \nabla_{\bar{1}} \bar{e}_2 \perp \bar{e}_1 \\ \nabla_1 \bar{e}_2 \perp \bar{e}_1 \\ \nabla_{\bar{2}} \bar{e}_2 \perp \bar{e}_1 \\ \nabla_2 \bar{e}_2 \perp \bar{e}_1 \end{cases}$$

Observe that $C \iff A, D \iff B$ by taking conjuation and (a) $\iff$ (c$'$), (b) $\iff$ (d$'$), (c) $\iff$ (a$'$), (d) $\iff$ (b$'$) by interchanging e_1 and e_2. Thus it suffices to show (c$'$), (d$'$), (c) and (d).

(c$'$):

$$d_{e_1} R_{2\bar{1}2\bar{1}} = (\nabla_1 R)_{2\bar{1}2\bar{1}} + R_{\nabla_1 2\bar{1}2\bar{1}} + R_{2\nabla_1\bar{1}2\bar{1}} + R_{2\bar{1}\nabla_2 2\bar{1}} + R_{2\bar{1}2\nabla_2\bar{1}}$$

and $d_{e_1} R_{2\bar{1}2\bar{1}} = 0, R_{\nabla_1 2\bar{1}2\bar{1}} = 0, R_{2\nabla_1\bar{1}2\bar{1}} = 0, R_{2\bar{1}\nabla_2 2\bar{1}} = 0, R_{2\bar{1}2\nabla_2\bar{1}}$ imply that $(\nabla_1 R)_{2\bar{1}2\bar{1}} = 0$. By the second Bianchi identity, $(\nabla_1 R)_{2\bar{1}2\bar{1}} = (\nabla_2 R)_{2\bar{1}1\bar{1}}$, we have $(\nabla_2 R)_{2\bar{1}1\bar{1}} = 0$.

$$d_{e_2} R_{2\bar{1}1\bar{1}} = (\nabla_2 R)_{2\bar{1}1\bar{1}} + R_{\nabla_2 2\bar{1}1\bar{1}} + R_{2\nabla_2\bar{1}1\bar{1}} + R_{2\bar{1}\nabla_2 1\bar{1}} + R_{2\bar{1}1\nabla_2\bar{1}}$$

and $d_{e_2} R_{2\bar{1}1\bar{1}} = 0, (\nabla_2 R)_{2\bar{1}1\bar{1}} = 0, R_{\nabla_2 2\bar{1}1\bar{1}} = 0, R_{2\bar{1}\nabla_2 1\bar{1}} = 0, R_{2\bar{1}1\nabla_2\bar{1}} = 0$ imply that $R_{2\nabla_2\bar{1}1\bar{1}} = 0$. That is to say $\nabla_2 e_2$ can not have a component in the direction of e_1 i.e. $\nabla_2 e_2 \perp e_1$. This is (c$'$).

(d$'$):

$$d_{e_1} R_{1\bar{1}2\bar{2}} = (\nabla_1 R)_{1\bar{1}2\bar{2}} + R_{\nabla_1 1\bar{1}2\bar{2}} + R_{1\nabla_1\bar{1}2\bar{2}} + R_{1\bar{1}\nabla_1 2\bar{2}} + R_{1\bar{1}2\nabla_1\bar{2}}$$

and $d_{e_1} R_{1\bar{1}2\bar{2}} = 0, R_{\nabla_1 1\bar{1}2\bar{2}} = 0, R_{1\nabla_1\bar{1}2\bar{2}} = 0, R_{1\bar{1}\nabla_1 2\bar{2}} = 0, R_{1\bar{1}2\nabla_1\bar{2}} = 0$ imply that $(\nabla_1 R)_{1\bar{1}2\bar{2}} = 0$. By the second Bianchi identity, $(\nabla_1 R)_{1\bar{1}2\bar{2}} = (\nabla_2 R)_{1\bar{1}1\bar{2}}$, we have $(\nabla_2 R)_{1\bar{1}1\bar{2}} = 0$.

$$d_{e_2} R_{1\bar{1}1\bar{2}} = (\nabla_2 R)_{1\bar{1}1\bar{2}} + R_{\nabla_2 1\bar{1}1\bar{2}} + R_{1\nabla_2\bar{1}1\bar{2}} + R_{1\bar{1}\nabla_2 1\bar{2}} + R_{1\bar{1}1\nabla_2\bar{2}}$$

and $d_{e_2}R_{1\bar{1}1\bar{2}} = 0, (\nabla_2 R)_{1\bar{1}1\bar{2}} = 0, R_{\nabla_2 1\bar{1}1\bar{2}} = 0, R_{1\nabla_2\bar{1}1\bar{2}} = 0, R_{1\bar{1}\nabla_2 1\bar{2}} = 0$ imply that $R_{1\bar{1}1\nabla_2\bar{2}} = 0$. That is $\nabla_2\bar{e}_2$ cannot have a component in the direction of $\bar{e}_1$, so $\nabla_2\bar{e}_2 \perp \bar{e}_1$, take conjugation and get $\nabla_{\bar{2}}e_2 \perp e_1$. This is (d′).

(c): $\langle e_1, e_2\rangle = 0 \Longrightarrow \langle \nabla_2 e_1, e_2\rangle + \langle e_1, \nabla_2 e_2\rangle = 0 \Longrightarrow \langle \nabla_2 e_1, e_2\rangle + 0 = 0 \Longrightarrow \nabla_2 e_1 \perp e_2$ and hence (c).

(d): $\langle e_1, e_2\rangle = 0 \Longrightarrow \langle \nabla_{\bar{2}} e_1, e_2\rangle + \langle e_1, \nabla_{\bar{2}} e_2\rangle = 0 \Longrightarrow \langle \nabla_{\bar{2}} e_1, e_2\rangle + 0 = 0 \Longrightarrow \nabla_{\bar{2}} e_1 \perp e_2$ and hence (d).

Therefore by de Rham's decomposition theorem, (ref. [Kobayashi] Chapter IV), saying that the M will be cross product of two submanifolds M' and M'' where M' and M'' are the maximal integral manifolds of T' and T'', respectively. Since $R_{1\bar{1}1\bar{1}}$ and $R_{2\bar{2}2\bar{2}}$ constants. Therefore the product structure are biholomorphic to $\Delta \times \Delta$. There is also a proof in [Zheng] (p. 243).

4. Chern-Number Inequality on Stable Vector Bundles with Hermitian-Einstein Metrics over Compact Complex Manifolds

We will consider a Chern-number inequality on stable vector bundles. The fact is that for any *stable* vector bundle over a compact Kähler manifold, there exists a Hermitian-Einstein metric of the stable bundle. Hence we will consider a stable vector bundle over a compact Kähler manifold and use its Hermitian-Einstein metric to show the Chern-number inequality. The proof is from [Siu].

Let M be a compact complex Hermitian manifold with Kähler form $\omega = \sqrt{-1}\sum_{i,j} g_{i\bar{j}} dz^i \wedge dz^j$. Let $E \longrightarrow M$ be a Hermitian vector bundle of rank r. Let $(\Theta_\alpha{}^\beta)$ be the curvature matrix of E. In coordinates we have

$$\Theta_\alpha^\beta = \sum_{i,j} \Theta_\alpha{}^\beta{}_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

where $\Theta_\alpha{}^\beta{}_{i\bar{j}} = \Theta_\alpha{}^\beta\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right)$. Let $\wedge\Theta$ be the contraction of Θ with the Kähler form, i.e.

$$(\wedge\Theta)_\alpha{}^\beta = \sum_{i,j} g^{i\bar{j}}\Theta_\alpha{}^\beta{}_{i\bar{j}}.$$

Definition 4.1 (Hermitian-Einstein metric). Let h be a Hermitian metric along the fibers of the holomorphic vector bundle E over a Kähler manifold M. h is called *Hermitian-Einstein* if the contraction of curvature is constant, that is,

$$\wedge\Theta = \lambda I$$

where λ is a constant and I is the identity endomorphism of E.

Let $E \longrightarrow M$ be a holomorphic vector bundle of rank r and M is a compact complex Kähler manifold of complex dimension n. For E' a subbundle of E of rank s, we define the *normalized first Chern number* by

$$\mu(E') = \frac{1}{s}c_1(E')[\omega]^{n-1}$$

where $[\omega]^{n-1}$ is the cohomology class defined by the Kähler form.

Definition 4.2 (Stable Bundles). The vector bundle E is said to be *(semi)stable* with respect to the Kähler class $[\omega]$ of M if

$$\mu(E') < (\leq)\mu(E)$$

for any proper subbundle E' of E.

Let $E \longrightarrow M$ be a stable bundle and h a Hermitian-Einstein metric on it. By definition, the first and second Chern class of the vector bundle E is given by

$$\begin{aligned} c_1(\Theta) &= \frac{1}{2\pi\sqrt{-1}}\sum_{\alpha}\Theta_{\alpha}{}^{\alpha} \\ c_2(\Theta) &= \left(\frac{1}{2\pi\sqrt{-1}}\right)^2\frac{1}{2}\sum_{\alpha,\beta}(\Theta_{\alpha}{}^{\alpha}\wedge\Theta_{\beta}{}^{\beta}-\Theta_{\alpha}{}^{\beta}\wedge\Theta_{\beta}{}^{\alpha}), \end{aligned}$$

where Θ is the curvature form of E. The inequality we want to show is as follows,

$$((r-1)c_1(\Theta)^2 - 2rc_2(\Theta)) \leq 0.$$

Proof. Since

$$(r-1)c_1(\Theta)^2 = (r-1)\left(\frac{\sqrt{-1}}{2\pi}\right)^2\sum_{\alpha,\beta}\left(\Theta_{\alpha}{}^{\alpha}\wedge\Theta_{\beta}{}^{\beta}\right);$$

$$2rc_2(E) = 2r\left(\frac{\sqrt{-1}}{2\pi}\right)^2\frac{1}{2}\sum_{\alpha,\beta}\left(\Theta_{\alpha}{}^{\alpha}\wedge\Theta_{\beta}{}^{\beta}-\Theta_{\alpha}{}^{\beta}\wedge\Theta_{\beta}{}^{\alpha}\right),$$

then

$$\begin{aligned} &(r-1)c_1(\Theta)^2 - 2rc_2(\Theta) \\ = \;&\frac{1}{(2\pi)^2}\sum_{\alpha,\beta}\Theta_{\alpha}{}^{\alpha}\wedge\Theta_{\beta}^{\beta} - r\frac{1}{(2\pi)^2}\sum_{\alpha}{}^{\beta}\wedge\Theta_{\beta}{}^{\alpha}. \end{aligned}$$

Thus,

$$
\begin{aligned}
& (r-1)c_1(\Theta)^2 - 2rc_2(\Theta) \\
= \ & \frac{-1}{(2\pi)^2}\Big(\sum_{\alpha,\beta} -{\Theta_\alpha}^\alpha \wedge {\Theta_\beta}^\beta + r\Theta_\alpha^\beta \wedge \Theta_\beta^\alpha\Big) \\
= \ & \frac{-1}{(2\pi)^2}\Big(-\sum_{\substack{\alpha,\beta,i,j\\ i\neq j}} {\Theta_\alpha}^\alpha{}_{i\bar{j}}{\Theta_\beta}^\beta{}_{j\bar{i}} + {\Theta_\alpha}^\alpha{}_{i\bar{i}}{\Theta_\beta}^\beta{}_{j\bar{j}} \\
& + \sum_{\substack{\alpha,\beta,i,j\\ i\neq j}} r{\Theta_\alpha}^\beta{}_{i\bar{j}}{\Theta_\beta}^\alpha{}_{j\bar{i}} - r{\Theta_\alpha}^\beta{}_{i\bar{i}}{\Theta_\beta}^\alpha{}_{j\bar{j}}\Big)dz^i \wedge d\bar{z}^j \wedge dz^j \wedge d\bar{z}^i \\
= \ & \frac{-1}{(2\pi)^2}\Big(\sum_{\substack{\alpha,\beta,i,j\\ i\neq j}} -{\Theta_\alpha}^\alpha{}_{i\bar{j}}{\Theta_\beta}^\beta{}_{j\bar{i}} + (\wedge Tr(\Theta))^2 \\
& + \sum_{\substack{\alpha,\beta,i,j\\ i\neq j}} r{\Theta_\alpha}^\beta{}_{i\bar{j}}{\Theta_\beta}^\alpha{}_{j\bar{i}} - rTr(\wedge\Theta)^2\Big)dz^i \wedge d\bar{z}^j \wedge dz^j \wedge d\bar{z}^i \\
= \ & \frac{-1}{(2\pi)^2}\Big(\frac{1}{2}\sum_{\alpha,\beta,i,j} |{\Theta_\alpha}^\alpha{}_{i\bar{j}} - {\Theta_\beta}^\beta{}_{i\bar{j}}|^2 + (\wedge Tr\Theta)^2 \\
& + r\sum_{\substack{\alpha,\beta,i,j,\\ i\neq j}} |{\Theta_\alpha}^\beta{}_{i\bar{j}}|^2 - rTr(\wedge\Theta)^2\Big)dz^i \wedge d\bar{z}^j \wedge dz^j \wedge d\bar{z}^i
\end{aligned}
$$

Claim: $(\wedge Tr\Theta)^2 = rTr(\wedge\Theta)^2$. Proof of claim: Using the condition of being Hermitian-Einstein we have $\wedge\Theta = \gamma I$ where $\gamma = \frac{1}{r}Tr \wedge \Theta$, then

$$(\wedge Tr\Theta)^2 - rTr(\wedge\Theta)^2 = (\wedge Tr\Theta)^2 - r \cdot r \cdot \gamma^2 = (\wedge Tr\Theta)^2 - r^2 \cdot \frac{1}{r^2}(Tr \wedge \Theta)^2 = 0.$$

By the claim, we have the Chern-number inequality of E

$$(r-1)c_1^2 - 2rc_2 = \int_M ((r-1)c_1(\Theta)^2 - 2rc_2(\Theta))[\omega]^{n-2} \leq 0.$$

□

CHAPTER 4

Euler-Poincaré Characteristics of Vector Bundles

The first part of this chapter is devoted to the preliminary on harmonic theory, in particular, the Hodge theorem which gives that the cohomology groups of compact manifolds are of finite dimension. In the second part of this chapter, Euler-Poincaré characteristics of vector bundles are defined. To obtain several formulas of Euler-Poincaré characteristics of subbundles, we will also study the divisors. The last part of the chapter gives a generalized version of index of a vector bundle and then its equivalence between the Euler-Poincaré characteristic for vector bundle. This can be viewed as a generalization of the Poincaré-Hopf theorem for differentiable manifolds. The main tools used in the proof are taken from the harmonic theory, including the Hodge decomposition, the hard Lefschetz theorem and the Lefschetz decomposition. The main references of this chapter are [Hir], [Voisin] and [Griffiths].

1. Dolbeault Cohomology, Harmonic Theory and Hodge Theorem

This section is on harmonic theory. We will show that the cohomology groups of a vector bundle over a compact complex manifold is finite dimensional.

1.1. Dolbeault Cohomology Groups. Let M be a complex manifold and E be a holomorphic vector bundle over it. Recall from Chapter 2 that $\Omega^{p,q}(E) = \Omega^{p,q}(M) \otimes \Omega^0(E)$ is the group of holomorphic (p,q)-forms on M with coefficients being holomorphic sections of E and the operator

$$\bar{\partial}_E : \Omega^{p,q}(E) \longrightarrow \Omega^{p,q+1}(E)$$

is defined as an extension of $\bar{\partial} : \Omega^{p,q}(M) \longrightarrow \Omega^{p,q+1}(M)$ by regarding holomorphic sections in E as coefficients. Let ω be a holomorphic (p,q)-form in $\Omega^{p,q}(E)$. If $\bar{\partial}_E \omega = 0$ then ω is called a *$\bar{\partial}_E$-closed (p,q)-form with coefficients in E*; If there exists a $(p,q-1)$-form $\eta \in \Omega^{p,q-1}(E)$ such that $\omega = \bar{\partial}_E \eta$, then ω is called an *$\bar{\partial}_E$-exact p-form.* Denote

$$Z^{p,q}(E) = \{\bar{\partial}_E\text{-closed } (p,q)\text{-forms in } \Omega^{p,q}(E)\};$$

$$B^{p,q}(E) = \{\bar{\partial}_E\text{-exact } (p,q)\text{-forms in } \Omega^{p,q}(E)\}.$$

From $d^2 = 0$ we have $\bar{\partial}^2 = 0$ and hence $\bar{\partial}_E^2 = 0$, thus we can define a quotient group as follows.

Definition 1.1 (Dolbeault Cohomology Group).

$$H^{p,q}_{Dol}(M,E) = \frac{Z^{p,q}(E)}{B^{p,q}(E)}$$

is called the (p,q)*-th Dolbeault cohomology group of* M *with coefficients in* E.

This definition can be understood in the way that $\Omega^{p,q}(E)$ becomes sheaf of germs of holomorphic sections of the vector bundle $E \otimes \wedge^p T^{1,0}M \otimes \wedge^q T^{0,1}M$ and $\bar{\partial}_E$ be the induce sheaf homomorphism. We will understand the Dolbeault cohomology group in this way and simply denote $\bar{\partial}_E$ by $\bar{\partial}$ in the following context. For any holomorphic vector bundle $V \longrightarrow M$, let $\Omega(V)$ also denote the sheaf of germs of holomorphic sections of V. Then the *Čech cohomology group of* M *with coefficients in* $\Omega(V)$ is denoted by $\check{\mathrm{H}}^*(M,V)$. Let $V = E \otimes \wedge^p T^{1,0}M$ and denote $\check{\mathrm{H}}^q(M, E \otimes \wedge^p T^{1,0}M)$ by $H^{p,q}(M,E)$, then we have the isomorphism between the Dolbeault cohomology group and the Čech cohomology group as follows:

THEOREM 1.1 (Dolbeault-Serre Isomorphism).

$$H^{p,q}(M,E) \cong H^{p,q}_{Dol}(M,E) \left(= \frac{Z^{p,q}(E)}{B^{p,q}(E)} \right).$$

PROOF. The long exact sequence of sheaves of germs of holomorphic sections of vector bundles over M,

$$\begin{aligned} &0 \longrightarrow \Omega(E \otimes \wedge^p T^{1,0}M) \xrightarrow{i} \Omega^{p,0}(E) \xrightarrow{\bar{\partial}} \Omega^{p,1}(E) \xrightarrow{\bar{\partial}} \\ &\cdots \xrightarrow{\bar{\partial}} \Omega^{p,q}(E) \xrightarrow{\bar{\partial}} \Omega^{p,q+1}(E) \xrightarrow{\bar{\partial}} \cdots, \end{aligned}$$

can be rewritten in the form of short exact sequences:

$$\begin{aligned} &0 \longrightarrow \Omega(E \otimes \wedge^p T^{1,0}M) \xrightarrow{i} \Omega^{p,0}(E) \xrightarrow{\bar{\partial}} Z^{p,1}(E) \longrightarrow 0; \\ &0 \longrightarrow Z^{p,1}(E) \xrightarrow{i} \Omega^{p,1}(E) \xrightarrow{\bar{\partial}} Z^{p,2}(E) \longrightarrow 0; \\ &\qquad\qquad\vdots \\ &0 \longrightarrow Z^{p,q}(E) \xrightarrow{i} \Omega^{p,q}(E) \xrightarrow{\bar{\partial}} Z^{p,q+1}(E) \longrightarrow 0. \end{aligned}$$

Each of the short exact sequence will induce a long exact sequence, partly written as follows:

$$\begin{aligned} &\cdots \longrightarrow H^{q-1}(M, Z^{p,1}(E)) \xrightarrow{\delta} H^q(M, \Omega(E \otimes \wedge^p T^{1,0}M)) \longrightarrow \cdots; \\ &\cdots \longrightarrow H^{q-2}(M, Z^{p,2}(E)) \xrightarrow{\delta} H^{q-1}(M, Z^{p,1}(E)) \longrightarrow \cdots; \\ &\qquad\qquad\vdots \\ &\cdots \longrightarrow H^0(M, Z^{p,q-1}(E)) \longrightarrow H^0(M, \Omega^{p,q-1}(E)) \longrightarrow \\ &\quad \longrightarrow H^0(M, Z^{p,q}(E)) \xrightarrow{\delta} H^1(M, Z^{p,q-1}(E)) \longrightarrow \cdots. \end{aligned}$$

where δ stands for the connection operator. By $\check{\mathrm{H}}^k(M, \Omega^{p,q}(E)) = 0$ for all $k > 0$ and all p, q ([Griffiths] p. 45.), then

$$\begin{aligned} & H^q(M, \Omega(E \otimes \wedge^p T^{1,0} M)) \\ \cong \;& H^{q-1}(M, \Omega(E \otimes Z^{p,1})) \\ \cong \;& H^{q-2}(M, Z^{p,2}(E)) \\ \vdots \;& \\ \cong \;& H^1(M, Z^{p,q-1}(E)) \\ \cong \;& \frac{H^0(M, Z^{p,q}(E))}{\bar{\partial} H^0(M, \Omega^{p,q-1}(E))} \\ \cong \;& \frac{\Gamma(M, Z^{p,q}(E))}{\bar{\partial}\Gamma(M, \Omega^{p,q-1}(E))} \cong \frac{Z^{p,q}(E)}{\bar{\partial}\Omega^{p,q-1}(E)}. \end{aligned}$$

□

1.2. Harmonic Forms and Harmonic Spaces. The proofs of the following important theorems, namely the Hodge theorem, the Hodge decomposition, the Lefschetz theorem and the Lefschetz decomposition, rely on the fact that we can represent the cohomology classes by a particular kind of differential forms which is closed and coclosed, namely the harmonic forms, provided that there is an Riemannian metric on M. With this representations, theory of harmonic forms, which is mostly from analysis using the L^2-metric, can be translated to the theorems mentioned above in terms of the normal cohomology classes. This subsection will define harmonic forms and harmonic spaces and will state a basic theorem, namely the Hodge theorem.

1.2.1. *Adjoint of d over a Compact Riemannian Manifolds.* Let M be a compact Riemannian manifold of dimension n with the Riemannian metric g. This metric first induces pointwisely an inner product $(\cdot,\cdot)_x : T_x^{*k} M \times T_x^{*k} M \longrightarrow \mathbb{R}$ in the following way: Let $\{dx^i\}$ of $T^*(M)_x$ be the dual of the basis $\{\frac{\partial}{\partial x_i}\}$ of $T_x M$, then $\{dx^I\}$ where $I \subset \{1, \ldots, n\}$ and $|I| = k$ forms an orthonormal basis for the inner product $(\cdot,\cdot)_x$. The pointwise inner product $(\cdot,\cdot)_x, x \in M$ induces further an L^2-metric $(\cdot,\cdot)_{L^2}$ on the space of $A^k(M, \mathbb{R})$ defined by the formula

$$(\xi, \eta)_{L^2} = \int_M (\xi(x), \eta(x))_x Vol(x)$$

for $\xi, \eta \in A^k(M, \mathbb{R})$ and $Vol(x)$ is the volume form with respect to g of M at point x. With respect to the L^2-metric, we can define the adjoint operator d^* of d by requiring $(\xi, d\eta)_{L^2} = (d^*\xi, \eta)_{L^2}$, for all $\xi \in A^{k+1}(M, \mathbb{R}), \eta \in A^k(M, \mathbb{R})$.

Proposition 1.1. *The adjoint d^* of d does exist.*

Proof. We will prove the existence by construction. Define the operator

$$* : A^k(M, \mathbb{R}) \longrightarrow A^{n-k}(M, \mathbb{R}) \tag{58}$$

by requiring that $(\psi(x), \eta(x))_x Vol(x) = \psi(x) \wedge *\eta(x)$, for all $\psi, \eta \in A^k(M), x \in M$. In local coordinates, for any k-form $\eta = \sum_I \eta_I dx^I$ where $I \subset$

$\{1,\ldots,n\}, |I| = k$,

$$*\eta = \sum_I \varepsilon_I dx^{I^C},$$

where $I^C = \{1,\ldots,n\} - I$ and ε_I denotes the sign of the permutation $(1,\ldots,n) \mapsto (I, I^C)$. Note that $**\eta = (-1)^q\eta$ and also $*^{-1} = (-1)^q *$. We now claim that the adjoint operator is given by $d^* = -*d*$. Indeed, for any $\psi \in A^{k-1}(M,\mathbb{R})$ and $\eta \in A^k(M,\mathbb{R})$,

$$\begin{aligned}
(\psi, d^*\eta) &= (d\psi,\eta)_{L^2} \\
&= \int_M (d\psi(z),\eta(z))_z Vol(M) \\
&= \int_M d\psi \wedge *\eta \\
&= (-1)^q \int_M \psi \wedge d*\eta + \int_M d(\psi\wedge *\eta) \\
&= (-1)^q \int_M \psi \wedge d*\eta + 0 \quad \text{(by Stokes' Theorem)} \\
&= -\int_M \psi \wedge **d*\eta \qquad \text{(by substituting } (-1)^k\eta = **\eta) \\
&= (\psi, -*d*\eta)_{L^2}.
\end{aligned}$$

Hence $d^* = -*d*$ and the existence is shown. □

1.2.2. *Adjoints of ∂ and $\bar{\partial}$ over a Compact Hermitian Manifold.* Let M be a compact Hermitian manifold M of complex dimension n endowed with a Hermitian metric h, we can induce an L_2-metric $(\cdot,\cdot)_{L^2}$ on the space of $\Omega^{p,q}(M)$. Let ω be the Kähler form of the Hermitian metric h. Under unitary frames we have pointwisely

$$\omega = \frac{\sqrt{-1}}{2}\sum_j dz^j \wedge d\bar{z}^j.$$

Induced from h, there is an induced inner product $(\cdot,\cdot)_z : T_z^{*(p,q)}M \times T_z^{*(p,q)}M \longrightarrow \mathbb{C}$ for each point $z \in M$ defined as follows: With respect to the basis $\{dz^I \wedge d\bar{z}^J\}_{|I|=p,|J|=q}$ where $I, J \subset \{1,\ldots,n\}$ of the space $T_z^{*(p,q)}M$, the induced inner product $(\cdot,\cdot)_z$ is defined in such a way that the basis is orthogonal and with norm $||dz^I \wedge d\bar{z}^J||^2 = 2^{p+q}$. Define the L^2-metric $(\cdot,\cdot)_{L^2}$ on the space $\Omega^{p,q}(M)$ by

$$(\psi,\eta)_{L^2} = \int_M (\psi(z),\eta(z))_z Vol(z)$$

for any $\psi,\eta \in \Omega^{p,q}(M)$ and the volume form $Vol(z)$ with respect to h at point $z \in M$ is given by $Vol(z) = \frac{\omega_z^n}{n!} = (-1)^{\frac{n(n-1)}{2}}(\frac{\sqrt{-1}}{2})^n dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n{}_z$. With respect to the L^2-metric $(\cdot,\cdot)_{L^2}$ on $\Omega^{p,q}(M)$, we can define the adjoints ∂^* and $\bar\partial^*$ of ∂ and $\bar\partial$ respectively by requiring

$$(\xi, \partial^*\eta)_{L^2} = (\partial\xi, \eta)_{L^2};$$
$$(\xi, \bar\partial^*\eta)_{L^2} = (\bar\partial\xi, \eta)_{L^2} \quad \text{for all } \xi \in \Omega^{p,q-1}(M), \eta \in \Omega^{p,q}(M).$$

PROPOSITION 1.2. *The adjoints ∂^* and $\bar\partial^*$ thus defined exists.*

PROOF. We will show the existence of $\bar\partial^*$ by construction and omit the proof of the existence of ∂^*. Define the operator

$$* : \Omega^{p,q}(M) \longrightarrow \Omega^{n-p,n-q}(M) \tag{59}$$

by requiring $(\xi(z), \eta(z))_z Vol(M) = \psi(z) \wedge *\eta(z)$, for all $\xi, \eta \in \Omega^{p,q}(M)$. In local coordinates, if $\eta \in \Omega^{p,q}(M)$ is written as $\eta = \sum_{I,J} \eta_{I\bar J} dz^I \wedge d\bar z^J$ where $I, J \subset \{1, \ldots, n\}$ and $|I| = p, |J| = q$, then

$$*\eta = 2^{p+q-n} \sum_{I,J} \varepsilon_{IJ} \bar\eta_{I\bar J} dz^{I^C} \wedge d\bar z^{J^C}$$

where $I^C = \{1, \ldots, n\} - I$ and $J^C = \{1, \ldots, n\} - J$ and ε_{IJ} denotes the sign of the permutation $(1, \ldots, n, 1', \ldots, n') \mapsto (I, J, I^C, J^C)$. Note also that $**\eta = (-1)^{p+q}\eta$. We now claim that the adjoint operator is given by $\bar\partial^* = - * \bar\partial *$. Indeed, for any $\xi \in \Omega^{p,q-1}(M)$ and $\eta \in \Omega^{p,q}(M)$,

$$\begin{aligned}
(\xi, \bar\partial^*\eta) &= (\bar\partial\xi, \eta)_{L^2} \\
&= \int_M \bar\partial\xi \wedge *\eta \\
&= (-1)^{p+q} \int_M \xi \wedge \bar\partial * \eta + \int_M \bar\partial(\xi \wedge *\eta) \\
&= (-1)^{p+q} \int_M \xi \wedge \bar\partial * \eta + \int_M d(\xi \wedge *\eta) \quad (\bar\partial = d \text{ on } (n, n-1)\text{-forms}) \\
&= (-1)^{p+q} \int_M \xi \wedge \bar\partial * \eta + 0 \quad (\text{by Stokes' Theorem}) \\
&= - \int_M \xi \wedge ** \bar\partial * \eta \quad (\text{by substituting } (-1)^{p+q}\eta = **\eta).
\end{aligned}$$

Hence $\bar\partial^* = - * \bar\partial *$ and the existence is shown. □

1.2.3. *Adjoints of ∂ and $\bar\partial$ over a Holomorphic Vector Bundle over a Compact Complex Manifold.* Let E be a holomorphic vector bundle over a compact complex manifold M of dimension n. We will generalize an L^2-metric on $\Omega^{p,q}(E)$. With respect to the open covering $\mathcal{U} = \{U_i\}$, if the transition functions of E are given by $f_{ij} : U_{ij} \longrightarrow GL(q, \mathbb{C})$, then the *dual bundle* $E^* \longrightarrow M$ is defined to be the vector bundle with transition functions $f_{ij}^{-1} : U_{ij} \longrightarrow GL(q, \mathbb{C})$ with respect to the same open covering $\mathcal{U}$ of M. Assume that the structure group $GL(q, \mathbb{C})$ of the vector bundle $E \longrightarrow M$ can be reduced to the unitary group $U(q)$. That is, there exist mappings $g_{ij} : U_{ij} \longrightarrow U(q)$ such that the image of $\{g_{ij}\}$ under the inclusion $H^2(M, U(q)) \hookrightarrow H^2(M, GL(q, \mathbb{C}))$ is equal to $\{f_{ij}\}$. Or equivalently, $\{f_{ij}/g_{ij}\}$ is a boundary, i.e. there exists

$\{h_i\} \in C^1(M, GL(q,\mathbb{C}))$ such that $f_{ij}/g_{ij} = h_j/h_i$ or $g_{ij} = h_i f_{ij} h_j^{-1}$. Let $g_i = \bar{h}_i^t h_i \in GL(q,\mathbb{C})$, and define a mapping

$$\Psi : E \longrightarrow E^*$$

by $\Psi(u,t) = (u, \overline{g_i(u)\cdot t})$, for $u \in U_i, t \in \mathbb{C}_q$. Ψ is called the *Hermitian anti-isomorphism defined by the reduction of structure groups.* Define the operators

$$\Xi : \Omega^{p,q}(E) \longrightarrow \Omega^{n-p,n-q}(E^*);$$
$$\tilde{\Xi} : \Omega^{r,s}(E^*) \longrightarrow \Omega^{n-r,n-s}(E)$$

by

$$\Xi = \Psi \otimes * : E \otimes \Omega^{p,q}(M) \longrightarrow E^* \otimes \Omega^{n-p,n-q}(M);$$
$$\tilde{\Xi} = \Psi^{-1} \otimes * : E^* \otimes \Omega^{r,s}(M) \longrightarrow E \otimes \Omega^{n-r,n-s}(M)$$

where the operator $*$ is defined as in (59). Define a product $\wedge : \Omega^{p,q}(E) \otimes \Omega^{r,s}(E^*) \longrightarrow \Omega^{p+r,q+s}(M)$ by requiring for any $(\xi,\eta) \mapsto \xi \wedge \eta$, the following conditions will be satisfied:

(i) $\bar{\partial}(\xi \wedge \eta) = \bar{\partial}\xi \wedge \eta + (-1)^{p+q}\xi \wedge \bar{\partial}\eta$;
(ii) $\xi \wedge \eta = (-1)^{(p+q)(r+s)}\eta \wedge \xi$.

Note that when E is the trivial line bundle 1, this product is the usual exterior product of differential forms. In particular, if $r = n-p, s = n-q$, the product is

$$\wedge : \Omega^{p,q}(E) \otimes \Omega^{n-p,n-q}(E^*) \longrightarrow \Omega^{n,n}(M).$$

We can define the induced L^2-metric as follows, Define $(\cdot,\cdot)_{L^2} : \Omega^{p,q}(E) \times \Omega^{p,q}(E) \longrightarrow \mathbb{C}$ by

$$(\alpha,\beta)_{L^2} = \int_M \alpha \wedge (\Xi\beta). \tag{60}$$

With respect to the L_2-metric of $\Omega^{p,q}(E)$, the adjoints $\bar{\partial}^*$ of the operator $\bar{\partial} : \Omega^{p,q}(E) \longrightarrow \Omega^{p,q+1}(E)$ are defined by requiring $(\xi, \bar{\partial}^*\eta)_{L^2} = (\bar{\partial}\xi, \eta)_{L^2}$, for all $\xi \in \Omega^{p,q-1}(E), \eta \in \Omega^{p,q}(E)$. The proof of the following proposition is similar to the proof of Proposition 1.2. We will omit it.

Proposition 1.3. *The adjoint $\bar{\partial}^*$ exists and is equal to $-\tilde{\Xi}\bar{\partial}\Xi$.*

1.2.4. *Hodge Theorem.* Define the *complex Laplace-Beltrami operator* $\Box : \Omega^{p,q}(E) \longrightarrow \Omega^{p,q}(E)$ by

$$\Box = \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*.$$

Note that the complex Laplace-Beltrami operator $\Box = \frac{\Delta_d}{2}$ where $\Delta_d = dd^* + d^*d$ is the *real Laplace operator.*

Definition 1.2 (Harmonic Form, Harmonic Space). A form $\eta \in \Omega^{p,q}(E)$ is said to be a *complex harmonic form* if $\Box\eta = 0$, the space

$$\mathbb{H}^{p,q}(M,E) = \{\eta \in \Omega^{p,q}(E) : \Box\eta = 0\}$$

is called the *space of complex harmonic (p,q)-forms.*

PROPOSITION 1.4. *For any $\eta \in \Omega^{p,q}(E)$, $\Box\eta = 0$ if and only if $\bar{\partial}^*\eta = 0$ and $\bar{\partial}\eta = 0$.*

PROOF. The proof of the "if" direction is obvious, we will show the "only if" direction. For any $\eta \in \Omega^{p,q}(M)$ such that $\Box\eta = 0$, we have

$$\begin{aligned} 0 &= (\Box\eta, \eta)_{L^2} \\ &= ((\bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*)\eta, \eta)_{L^2} \\ &= (\bar{\partial}^*\bar{\partial}\eta, \eta)_{L^2} + (\bar{\partial}\bar{\partial}^*\eta, \eta)_{L^2} = (\bar{\partial}\eta, \bar{\partial}\eta)_{L^2} + (\bar{\partial}^*\eta, \bar{\partial}^*\eta)_{L^2}. \end{aligned}$$

That is, $\bar{\partial}\eta = 0$ and $\bar{\partial}^*\eta = 0$ for any $\eta \in \Omega^{p,q}(E)$. □

The proof (ref. [Griffiths] p. 85.) of the Hodge theorem is by analysis. As an corollary, we also have the Kodaira vanishing theorem.

THEOREM 1.2 (Hodge Theorem). *For M be a compact complex manifold and E be a vector bundle over it. Then*

$$\Omega^{p,q}(E) = \bar{\partial}\Omega^{p,q-1}(E) \oplus \bar{\partial}^*\Omega^{p,q+1}(E) \oplus \mathbb{H}^{p,q}(M,E).$$

Thus $Z^{p,q}(E) = \bar{\partial}\Omega^{p,q-1} \oplus \mathbb{H}^{p,q}(M,E)$ and hence

$$\mathbb{H}^{p,q}(M,E) \cong Z^{p,q}(E)/\bar{\partial}\Omega^{p,q-1}(E).$$

Therefore,

$$H^{p,q}(M,E) \cong \mathbb{H}^{p,q}(M,E).$$

and further $\mathbb{H}^{p,q}(M,E)$ is finite dimensional.

THEOREM 1.3 (Kodaira Vanishing Theorem). *$H^{p,q}(M,E)$ is of finite dimensional. If $p > n$ or $q > n$ then $H^{p,q}(M,E) = 0$.*

2. Euler-Poincaré Characteristics of Vector Bundles

We will first give as preliminary the definition of the divisor and the divisor line bundle. Then define the Euler-Poincaré characteristics of vector bundles. For the needs of proving the Riemann-Roch theorem, this section will also define the generalized Euler characteristic χ_y and the virtual generalized Euler characteristic of virtual submanifolds which can be explained as intersection of divisors.

2.1. Divisors. One way of constructing submanifold S of a complex manifold M is to define it as a zero set of certain holomorphic function(s). The same idea lies in the definition of divisor. Let V be an *analytic hypersurface* of a complex manifold M. That is, V locally is the zero set of one single holomorphic function unique up to a non-zero holomorphic function on M. In other words, there exists an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of M such that the function f_i defined on each U_i.

$$S = \cup_{i\in I} \text{ zero set of } f_i$$

as a subset of U_i. This is an example of divisor, which we will define immediately. The analytic hypersurface V is called *irreducible* if V cannot be decomposed as disjoint union of two other analytic hypersurfaces S_1 and S_2 of M. Thus we can write $V = \cup_j S_j$ where each S_j is an irreducible analytic hypersurface.

Definition 2.1 (Divisor). Define a *divisor* S of M to be the formal linear combination

$$S = \sum_j a_j V_j$$

where $a_j \in \mathbb{Z}$ and V_j are irreducible analytic hypersurface.

Since analytic hypersurfaces are defined by the holomorphic functions which are unique up to non-zero functions locally, we can redefine a anaylitic hypersurface by given the functions adding certain compatible conditions. Furthermore, to add the information of degree of the meromorphic function, we add the degree as coefficients of a zero set. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of M, and $\{f_i\}_{i\in I}$ be a family of meromorphic functions such that f_i is defined on U_i and further, $\dfrac{f_i}{f_j}$ has neither zeros nor poles on the intersection U_{ij}. Observe that the union of the zero sets adding degree of any f_i on U_i gives a divisor in an unambiguous way. The set of such $\{f_i\}$ satisfying the above compatible condition with respect certain open covering $\mathcal{U}$ can also be viewed as a definition of *divisor*. The problem is that two such systems of functions may give the same divisor. We may revise the second definition of divisors by the language of sheaves.

Since changing the function f_i by multiplying a homeomorphic function h_i without zeros or poles on M, that is a holomorphic function without zeros on M for each i, gives the same divisor. Instead of defining the divisor by $\{f_i\}$ on $\{U_i\}$, we can define it by family of equivalence classes $\{[f_i]\}$ with representatives f_i's, with respect to $\{U_i\}$. Here the equivalence relation is given by $f_i \sim f_i h_i$ where h_i is a holomorphic function with no zeros on M. This way of defining divisors turns out to solve the problem above. In the language of sheaves, let $\mathfrak{G}$ be the sheaf of germs of locally meromorphic, not identically zero functions, C^*_ω be the sheaf of germs of nowhere zero holomorphic functions. Since C^*_ω is a subsheaf of $\mathfrak{G}$, then we can define the quotient sheaf $\mathfrak{D}$ by $\mathfrak{G}/C^*_\omega$. The revised version of definition of divisor is nothing but sections of the sheaf $\mathfrak{D}$ or equivalently elements in the cohomology group $H^0(M, \mathfrak{D})$.

From the short exact sequence,

$$0 \longrightarrow C^*_\omega \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{D} \longrightarrow 0,$$

we have the corresponding long exact sequence,

$$\cdots \longrightarrow H^0(M, \mathfrak{G}) \xrightarrow{h} H^0(M, \mathfrak{D}) \xrightarrow{\delta_0{}^*} H^1(M, C_\omega{}^*) \cdots .$$

Note that the $H^1(M, C_\omega{}^n)$ is the Picard group which is isomorphic to the group of isomorphic classes of line bundles on M. Therefore, each divisor $D \in H^0(M, \mathfrak{D})$ determines an element $\delta_0{}^*(D)$ in $H^1(M, C_\omega{}^n)$ and hence determine a line bundle $L(D)$ up to an isomorphism. We call $L(D)$ the *associated line bundle of the divisor D*.

Elements in $H^0(M, \mathfrak{G})$ is sections of the sheaf $\mathfrak{G}$ that is *global* meromorphic functions on M. For any such global meromorphic function f in $\Gamma(M, \mathfrak{G})$ we call the image $h(f)$ a *divisor of the global meromorphic function* f. The set $h(\Gamma(M, \mathfrak{G}))$ is isomorphic to a subgroup of the group $H^1(M, C_\omega{}^*)$. Similarly, $h(f) \in h(\Gamma(M, \mathfrak{G}))$ determines a unique line bundle, $L(f)$ over M, up to an isomorphism. Let us examine the property of this kind of line bundles. f is a global meromorphic function implies $f \times C_\omega{}^*$ is also global, $\delta_0{}^*\{f \times C_{\omega\alpha}^*\} = \{f \times C_{\omega\beta}^*/f \times C_{\omega\alpha}^*\} = id_{\alpha\beta}$ on $U_{\alpha\beta}$ thus the resulting line bundle is trivial.

2.2. Euler-Poincaré Characteristics and the Generalized Euler-Poincaré Characteristics. In the following context, M will be a compact complex manifold of complex dimension n and E will be a holomorphic vector bundle over it.

2.2.1. *χ and χ_y of Vector Bundles.* A sheaf $\mathfrak{S}$ of K-modules over a topological space X is *of type (F)* if $H^q(X, \mathfrak{S})$ are finite dimensional and $dimH^q(X, \mathfrak{S}) = 0$ for all but a finite number of $q \geq 0$.

Definition 2.2 (Euler-Poincaré Characteristic). For any type (F) sheaf $\mathfrak{G}$ over a differentiable manifold V, the *Euler characteristic* of $\mathfrak{G}$ is defined by

$$\chi(V, \mathfrak{G}) = \sum_{i=0}^{\infty}(-1)^i dimH^i(V, \mathfrak{G}).$$

For $E \longrightarrow M$, the holomorphic vector bundle over the compact complex n dimensional manifold, let $\Omega(E)$ denote the sheaf of germs of holomorphic sections of $E \longrightarrow M$, then the *Euler-Poincaré characteristic of* E is defined by

$$\chi(M, E) = \sum_{i=0}^{\infty}(-1)^i dimH^i(M, \Omega(E)).$$

The fact that $H^i(M, \Omega(E))$ are finite dimensional and are zero whenever $i > n$ if M is compact as stated by the Kodaira vanishing theorem, gives well-definedness of the Euler-Poincaré characteristics of E.

Definition 2.3 (χ_y-Characteristic). Let

$$\chi^p(M, E) = \chi(M, E \otimes \Omega^{p,0}(M)) = \sum_{q=0}^{n}(-1)^q h^{p,q}(M, E),$$

where $h^{p,q}(M, E) = dimH^q(M, E \otimes \Omega^{p,0}(M)$, then the *$\chi_y$-characteristic of* E is defined by

$$\chi_y(M, E) = \sum_{p=0}^{n} \chi^p(M, E)y^p.$$

Consider tangent bundle TM of M, we get the corresponding definition.

Definition 2.4 (χ_y-Genus). If

$$\chi^p(M) = \chi(M, \Omega^{p,0}(M)) = \sum_{q=0}^{n} (-1)^q h^{p,q}(M)$$

where $h^{p,q}(M) = dim H^q(M, \Omega^{p,0}(M))$, then the χ_y-*genus* of M is defined by

$$\chi_y(M) = \sum_{p=0}^{n} \chi^p(M) y^p.$$

The *arithmetic-genus* of M is defined to be

$$\chi(M) = \chi_0(M) = \chi^0(M).$$

The usual Euler-characteristic of M equals $\chi_{-1}(M)$.

2.2.2. *Some Properties of χ and χ_y.*

Proposition 2.1. *If $0 \longrightarrow \mathfrak{G}' \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{G} \longrightarrow 0$ is an exact sequence of sheaves over a differentiable manifold V, such that each sheaf is of type (F), i.e. $dim H^i(V, \mathfrak{G}(\mathfrak{G}')) = 0$ and $dim H^i(V, \mathfrak{G}(\mathfrak{G}'')) = 0$ for i bigger than certain integer N. Then,*

$$\chi(V, \mathfrak{G}) = \chi(V, \mathfrak{G}') + \chi(V, \mathfrak{G}'').$$

Proof. The following sequence is the long exact sequence induced from the short exact sequence in the assumption:

$$\begin{array}{lllll} 0 \longrightarrow & H^0(\mathfrak{G}') \xrightarrow{i_0} H^0(\mathfrak{G}) \xrightarrow{j_0} & H^0(\mathfrak{G}'') \xrightarrow{\delta_0^*} \\ \longrightarrow & H^1(\mathfrak{G}') \xrightarrow{i_1} H^1(\mathfrak{G}) \xrightarrow{j_1} & H^1(\mathfrak{G}'') \xrightarrow{\delta_1^*} \\ \longrightarrow & H^2(\mathfrak{G}') \xrightarrow{i_2} H^2(\mathfrak{G}) \xrightarrow{j_2} & H^2(\mathfrak{G}'') \xrightarrow{\delta_2^*} \cdots \\ \cdots \longrightarrow & H^p(\mathfrak{G}') \xrightarrow{i_p} H^p(\mathfrak{G}) \xrightarrow{j_p} & H^p(\mathfrak{G}'') \xrightarrow{\delta_p^*} \cdots . \end{array}$$

The Euler-Poincaré characteristic is

$$\chi(\mathfrak{G}') = \sum_p (-1)^p a_p, \quad \text{where} \quad a_p = dim H^p(\mathfrak{G}');$$

$$\chi(\mathfrak{G}) = \sum_p (-1)^p b_p, \quad \text{where} \quad b_p = dim H^p(\mathfrak{G});$$

$$\chi(\mathfrak{G}'') = \sum_p (-1)^p c_p, \quad \text{where} \quad c_p = dim H^p(\mathfrak{G}'').$$

Note that we have the following relations:

$$\begin{array}{lll} a_p & = & dim(Im i_p) + dim(Ker i_p); \\ b_p & = & dim(Im j_p) + dim(Ker j_p); \\ c_p & = & dim(Im \delta_p^*) + dim(Ker \delta_p^*). \end{array}$$

Thus $a_p+c_p = dim(Im\delta_p^*)+b_p+dim(Keri_p)$ by using the condition of exactness of the long exact sequence, $Im(i_p) = Ker(j_p)$ and $Im(j_p) = Ker(\delta_p^*)$ for all p. Consider

$$\begin{aligned} a_0 + c_0 &= dim(Im\delta_0) + b_0 + dim(Keri_0) \\ -(a_1 + c_1) &= -dim(Im\delta_1^*) - b_1 - dim(Keri_1) \\ a_2 + c_2 &= dim(Im\delta_2^*) + b_2 + dim(Keri_2) \\ &\vdots \\ (-1)^p(a_p + c_p) &= (-1)^p dim(Im\delta_p^*) + (-1)^p b_p + (-1)^p dim(Keri_p) \\ &\vdots \end{aligned}$$

and take summation of the sequence of identities and using again the condition of exactness to cancel the upper left term and the lower right term in adjacent lines, we have $\sum_p (-1)^p a_p + c_p = \sum_p (-1)^p b_p$, i.e. $\chi(\mathfrak{G}) = \chi(\mathfrak{G}') + \chi(\mathfrak{G}'')$. □

We will state without proof the following proposition.

PROPOSITION 2.2. *If* $0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$ *is a short exact sequence of holomorphic vector bundles over the compact complex manifold* M*, then the following sequence of sheaves of germs of holomorphic sections of vector bundles over* M *is also exact:*

$$0 \longrightarrow \Omega(E') \longrightarrow \Omega(E) \longrightarrow \Omega(E'') \longrightarrow 0.$$

The following theorem will be used in the proof of Riemann-Roch theorem of vector bundles.

THEOREM 2.1. *Let* E *be a holomorphic vector bundle of rank* q *over the compact complex manifold* M*, and suppose that the structure group* $GL(q, \mathbb{C})$ *of* E *can be reduced to the group* $\Delta(q, \mathbb{C})$*. Let* $A_1, \ldots, A_q$ *be the associated diagonal* $\mathbb{C}^*$*-bundles of* E*, then*

$$\chi(M, E) = \chi(M, A_1) + \chi(M, A_2) + \cdots + \chi(M, A_q).$$

PROOF. The theorem is true for $q = 1$. Prove by induction, we may assume the theorem is true for $(q-1)$. Consider the exact sequence

$$0 \longrightarrow A_1 \longrightarrow E \longrightarrow E/A_1 \longrightarrow 0.$$

By Proposition 2.2, the following sequence of sheaves over M is also exact

$$0 \longrightarrow \Omega(A_1) \longrightarrow \Omega(E) \longrightarrow \Omega(E/A_1) \longrightarrow 0.$$

By Proposition 2.1, we have $\chi(M, E) = \chi(M, A_1) + \chi(M, E/A_1)$. The vector bundle E/A_1 admits $\Delta(q-1, \mathbb{C})$ as structure group and with $A_2, \ldots, A_q$ as the associated $(q-1)$ $\mathbb{C}^*$-bundles. The induction assumption implies that $\chi(M, E/A_1) = \chi(M, A_2) + \chi(M, A_3) + \cdots + \chi(M, A_q)$. Hence

$$\begin{aligned} \chi(M, E) &= \chi(M, A_1) + \chi(M, W/A_1) \\ &= \chi(M, A_1) + \chi(M, A_2) + \cdots + \chi(M, A_q). \end{aligned}$$

□

2.2.3. *χ and χ_y of Divisor Line Bundles.* Suppose $E \longrightarrow M$ is a complex holomorphic vector bundle over a complex manifold M. Let $S \subset M$ be a *non singular divisor*, that is there exists an open covering $\mathcal{U} = \{U_i\}$ of M such that $S \cap U_i$ is given by an equation $s_i = 0$, where s_i is a holomorphic function defined on U_i with non-zero partial derivatives at each point $x \in U_i \cap S$. Let $L(S)$ be the associated line bundle on M with transition functions $\{s_{ij} = \frac{s_i}{s_j}\}$. Recall that the maps $s_i : U_i \longrightarrow \mathbb{C}$ defines a global section s of the line bundle $L(S)$, which is zero at all points in S and nonzero elsewhere. Define the vector bundle

$$(E \otimes \{S\})_S \longrightarrow E$$

to be the restriction of the vector bundle $E \otimes \{S\} \longrightarrow M$ onto S. Let $\Omega((E \otimes \{S\})_S)$ be the sheaf of germs of holomorphic sections of $E \otimes \{S\}_S$ over S. Define $\hat{\Omega}((E \otimes \{S\})_S)$ to be the sheaf over M to be the extension by zero of the sheaf $(\Omega((E \otimes \{S\})_S)$ from S to M in such a way that the germs in $\hat{\Omega}$ equal to the germs in Ω on S and equals to zero elsewhere.

PROPOSITION 2.3. *The following sequence of sheaves of locally holomorphic sections of vector bundles over M is exact:*

$$0 \longrightarrow \Omega(E) \xrightarrow{h_1} \Omega(E \otimes \{S\}) \xrightarrow{h_2} \hat{\Omega}((E \otimes \{S\})_S) \longrightarrow 0, \tag{61}$$

where h_1 and h_2 are defined in the canonical way.

PROOF. (i) h_1 is injective: For any $s' \in \Omega(E)$, the $h_1(s')$ is defined to be $s' \otimes s \in \Omega(E \otimes \{S\})$. Since s_i is not identically zero on any open set U_i, then the map h_1 is injective.

(ii) h_2 is surjective: Let $r : \Omega(E \otimes \{S\}) \longrightarrow \Omega((E \otimes \{S\})_S)$ be the restriction of germs on M to germs on S and $h : \Omega((E \otimes \{S\})_S) \longrightarrow \hat{\Omega}(E \otimes \{S\})$ be the zero extension. $h_2 : \Omega(E \otimes \{S\}) \longrightarrow \hat{\Omega}((E \otimes \{S\})_S)$ is defined by $h_2 = h \circ r$. Hence h_2 is surjective.

(iii) $Imh_1 \subset Kerh_2$: For any $s' \times s \in Imh_1$,

$$h_2(s' \otimes s) = h \circ r(s' \otimes s) = h(s' \otimes s|_S) = h(0) = 0.$$

(iv) $Kerh_2 \subset Imh_1$. Since elements in $\hat{\Omega}((E \otimes \{S\})_S)$ are in the form $h(s' \otimes s'')$ where $s' \in \Omega(E)$ and $s'' \in \Omega(\{S\}$. Write $s'(x) = (f_1(x), \ldots, f_q(x))$ and $s''(x) = g(x)$ for all $x \in S$. Then $h(s' \otimes s'') = 0$ implies that $s'(x) \otimes s''(x) = 0$ for $x \in S$. That is $(f_1(x)g(x), f_2(x)g(x), \ldots, f_q(x)g(x)) = (0, 0, \ldots, 0)$. This is equivalent to $f_i(x) = 0$ for all $i = 1, \ldots, q$ or $s(x) = 0$. Since either of the two conditions gives $s'' = s$, hence $Kerh_2 \subset Imh_1$. Exactness of (61) is shown. □

COROLLARY 2.4.

$$\chi(M, E) = \chi(M, E \otimes \{S\}^{-1}) + \chi(S, E). \tag{62}$$

PROOF. By replacing E in the sequence (61) by $E \otimes \{S\}^{-1}$, we have the following exact sequence, $0 \longrightarrow \Omega(E \otimes \{S\}^{-1}) \longrightarrow \Omega(E) \longrightarrow \hat{\Omega}(E_S) \longrightarrow 0$. By Proposition 2.1, we have $\chi(M, E) = \chi(M, E \otimes \{S\}^{-1}) + \chi(S, E)$. □

COROLLARY 2.5.

$$\chi^p(M, E) = \chi^p(M, E \otimes \{S\}^{-1}) + \chi(S, E \otimes \Omega^{p,0}(M)). \tag{63}$$

PROOF. By replacing E in identity (62) by $E \otimes \Omega^{p,0}(M)$, we have $\chi(M, E \otimes \Omega^{p,0}(M)) = \chi(M, E \otimes \Omega^{p,0}(M)) \otimes \{S\}^{-1}) + \chi(S, E \otimes \Omega^{p,0}(M))$. That is $\chi^p(M, E) = \chi^p(M, E \otimes \{S\}^{-1}) + \chi(S, E \otimes \Omega^{p,0}(M))$. □

LEMMA 2.6. *The sequence of vector bundles over S is exact:*

$$0 \longrightarrow \Omega^{p-1,0}(S) \otimes \{S\}_S^{-1} \longrightarrow \Omega^{p,0}(M)_S) \longrightarrow \Omega^{p,0}(S) \longrightarrow 0.$$

PROOF. Refer to [Hir] p. 131. □

By Lemma 2.6, we have the following exact sequence:

$$0 \longrightarrow E \otimes \Omega^{p-1,0}(S) \otimes \{S\}_S^{-1} \longrightarrow E \otimes \Omega^{p,0}(M)_S) \longrightarrow E \otimes \Omega^{p,0}(S) \longrightarrow 0.$$

Thus by Proposition 2.1 again, we have $\chi(S, E \otimes \Omega^{p,0}(M)_S) = \chi(S, E \otimes \Omega^{p-1,0}(S) \otimes \{S\}_S^{-1}) + \chi(S, E \otimes \Omega^{p,0}(S))$. This is equivalent to $\chi(S, E \otimes \Omega^{p,0}(M)) = \chi^{p-1}(S, E \otimes \{S\}_S^{-1}) + \chi^p(S, E)$, where $\chi^p(M, E)$ denotes $\chi(M, E \otimes \Omega^{p,0}(M))$ for all vector bundle $E \longrightarrow M$ and any $p \geq 0$. Compare with identity (63), we have the "four term formula":

THEOREM 2.2 (Four Term Formula). *The four term formula is given by*

$$\chi^p(M, E) = \chi^p(M, E \otimes \{S\}^{-1}) + \chi^p(S, E) + \chi^{p-1}(S, E \otimes \{S\}^{-1}). \tag{64}$$

Thus, we have

$$\begin{aligned}\sum_p \chi^p(M, E) y^p &= \sum_p \chi^p(M, E \otimes \{S\}^{-1}) y^p + \sum_p \chi^p(S, E) y^p \\ &\quad + \sum_p \chi^{p-1}(S, E \otimes \{S\}^{-1}) y^p,\end{aligned}$$

that is,

$$\chi_y^p(M, E) = \chi_y^p(M, E \otimes \{S\}^{-1}) + \chi_y^p(S, E) + \chi_y^{p-1}(S, E \otimes \{S\}^{-1}). \tag{65}$$

We can expressed any $\chi^p(S, E)$ in terms of $\chi^q(M, A)$ where A is some line bundle on M by using the four term formula (64) inductively. The resulting formula is

$$\chi^p(S, E) = \sum_{i=0}^{p} (-1)^i (\chi^{p-i}(M, E \otimes \{S\}^{-i}) - \chi^{p-i}(M, E \otimes \{S\}^{-(i+1)})). \tag{66}$$

PROPOSITION 2.7.

$$\chi_y(S, E) = \sum_{i=0}^{\infty} (-y)^i (\chi_y(M, E \otimes \{S\}^{-i}) - \chi_y(M, E \otimes \mathcal{S}\}^{-(i+1)})). \tag{67}$$

PROOF. By the formula (66),

$$\begin{aligned}\chi_y(S,E) &= \sum_{p=0}^{\infty}\chi^p(S,E)y^p \\ &= \sum_{p=0}^{\infty}(\sum_{i=0}^{p}(-1)^i(\chi^{p-i}(M,E\otimes\{S\}^{-i}) \\ &\quad -\chi^{p-i}(M,E\otimes\{S\}^{-(i+1)}))y^p\end{aligned}$$

On the left hand side,

$$\begin{aligned}&\sum_{i=0}^{\infty}(-y)^i(\chi_y(M,E\otimes\{S\}^{-i})-\chi_y(M,E\otimes\{S\}^{-(i+1)})) \\ =\ &\sum_{i=0}^{\infty}\sum_{p=0}^{\infty}(-1)^i(\chi^p(M,E\otimes\{S\}^{-i})y^{p+i}-\chi^p(M,E\otimes\{S\}^{-(i+1)})y^{p+i}) \\ \overset{p=p-i}{=}\ &\sum_{i=0}^{\infty}\sum_{p=i}^{\infty}(-1)^i(\chi^{p-i}(M,E\otimes\{S\}^{-i})y^p-\chi^{p-i}(M,E\otimes\{S\}^{-(i+1)})y^p) \\ =\ &\sum_{p=0}^{\infty}\sum_{i=0}^{p}(-1)^i(\chi^{p-i}(M,E\otimes\{S\}^{-i})y^p-\chi^{p-i}(M,E\otimes\{S\}^{-(i+1)})y^p).\end{aligned}$$

Therefore, the two sides coincide and the proposition is proved. □

2.3. Virtual χ_y-Characteristics.

2.3.1. *Preliminary on Algebra.* Let G be an extension ring over $\mathbb{Z}$ and y be an indeterminate. Let $G\{y\}$ be the ring of formal power series with coefficients in G and let $\mathbb{Z}\{y\}$ be the ring of formal power series with coefficients in $\mathbb{Z}$. Hence $\mathbb{Z}\{y\}$ is a subring of $G\{y\}$. If we regard both $G\{y\}$ and $\mathbb{Z}\{y\}$ as $\mathbb{Z}\{y\}$-modules, then a $\mathbb{Z}\{y\}$-homomorphism

$$h: G\{y\}\longrightarrow\mathbb{Z}\{y\}$$

is called a *d-homomorphism*. By definition, for h a d-homomorphism, it will satisfy:

(i) $h(u+v)=h(u)+h(v),$ for $u,v\in G\{y\}$;
(ii) $h(uv)=uh(v),$ for $u\in\mathbb{Z}\{y\}, v\in G\{y\}$.

LEMMA 2.8. *Let $h_0: G\longrightarrow\mathbb{Z}\{y\}$ be a homomorphism, then there exist a unique d-homomorphism $h: G\{y\}\longrightarrow\mathbb{Z}\{y\}$ such that $h|_G=h_0$.*

PROOF. Existence: For any $z=\sum a_iy^i\in G\{y\}, a_i\in G$, define $h(z)=\sum h_0(a_i)y^i$. Uniqueness: Suppose there are two such d-homomorphisms $h,h':G\{y\}\longrightarrow\mathbb{Z}\{y\}$ such that $h|_G=h'|_G=h_0$, then for any $z=\sum a_iy^i\in$

$G\{y\}, a_i \in G$,

$$h(z) = h(\sum a_i y^i) = \sum h(a_i) y^i = \sum h_0(a_i) y^i,$$
$$h'(x) = h'(\sum a_i y^i) = \sum h'(a_i) y^i = \sum h_0(a_i) y^i.$$

This implies $h = h'$. □

Given a d-homomorphism $h : G\{y\} \longrightarrow \mathbb{Z}\{y\}$ and a fixed power series $t = t(y) \in G\{y\}$, then we can get a new d-homomorphism $h_t : G\{y\} \longrightarrow \mathbb{Z}\{y\}$ defined by $h_t(u) = h(tu)$, for all $u \in G\{y\}$. This is all right since composition of d-homomorphisms is again a d-homomorphism.

2.3.2. *Virtual χ_y-Characteristics.* Now choose the extension ring over the ring $\mathbb{Z}$ to be $G = \mathbb{Z}\{f_1, \dots, f_r, f_1^{-1}, \dots, f_r^{-1}, w\}$ where $f_1, \dots, f_r, w$ are indeterminates and f_i^{-1} is the inverse of f_i, $i = 1, \dots, r$. The products

$$\{w^\mu f_1^{\lambda_1} f_2^{\lambda^2} \dots f_r^{\lambda_r}\},$$

where $\mu, \lambda_1, \dots, \lambda_r$ are all integers, form an additive basis of G. In general, suppose H is the assignment that assigns each product $w^\mu f_1^{\lambda_1} f_2^{\lambda^2} \dots f_r^{\lambda_r}$ an element, say $P^{\mu,\lambda_1,\dots,\lambda_r}(y)$ in $\mathbb{Z}\{y\}$, then there exists a unique homomorphism $h_0 : G \longrightarrow \mathbb{Z}\{y\}$ defined by

$$h_0(\sum_{\mu,\lambda_1,\dots,\lambda_r} a_{\mu,\lambda_1,\dots,\lambda_r} w^\mu f_1^{\lambda_1} f_2^{\lambda^2} \dots f_r^{\lambda_r}) = \sum_{\mu,\lambda_1,\dots,\lambda_r} a_{\mu,\lambda_1,\dots,\lambda_r} P^{\mu,\lambda_1,\dots,\lambda_r}(y)$$

where $a_{\mu,\lambda_1,\dots,\lambda_r} \in \mathbb{Z}$. Further by Lemma 2.8, there is a unique d-homomorphism $h : G\{y\} \longrightarrow \mathbb{Z}\{y\}$ determined by h_0.

In particular, let M be a compact complex manifold and $E \longrightarrow M$ a complex holomorphic vector bundle. Suppose $F_1, \dots, F_r$ are complex holomorphic line bundles over M. Let H and $\hat{H}$ be two assignments such that associate each product of G an element in $\mathbb{Z}\{y\}$ by

$$\begin{aligned} H(w^\mu f_1^{\lambda_1} f_2^{\lambda^2} \dots f_r^{\lambda_r}) &= \chi(M, E^\mu \otimes F_1^{\lambda_1} \otimes \dots \otimes F_r^{\lambda_r}) \in \mathbb{Z} \subset \mathbb{Z}\{y\}, \\ H(1) &= \chi(M); \end{aligned}$$

$$\begin{aligned} \hat{H}(w^\mu f_1^{\lambda_1} f_2^{\lambda_2} \dots f_r^{\lambda_r}) &= \chi_y(M, E^\mu \otimes F_1^{\lambda_1} \otimes \dots \otimes F_r^{\lambda_r}) \in \mathbb{Z}\{y\}, \\ \hat{H}(1) &= \chi_y(M). \end{aligned}$$

The two assignments induce two unique homomorphisms $h_0, \hat{h}_0 : G \longrightarrow \mathbb{Z}\{y\}$ respectively, and then two unique d-homomorphisms $h, \hat{h} : G\{y\} \longrightarrow \mathbb{Z}\{y\}$ respectively, as the way given above. In the following context, we will first find the formula of virtual χ_y-characteristic for only one cohomology class, which can be determined by a divisor S. Then generalize the formula to get the definition of virtual χ_y-characteristic of r cohomology classes, $r \geq 1$. Let S be a non-singular divisor of M, the given vector bundle can be restricted to S. Thus, we have the assignment

$$\begin{aligned} H_S(w^\mu f_1^{\lambda_1} f_2^{\lambda^2} \dots f_r^{\lambda_r}) &= \chi(S, E^\mu \otimes F_1^{\lambda_1} \otimes \dots \otimes F_r^{\lambda_r}) \in \mathbb{Z} \subset \mathbb{Z}\{y\}, \\ H_S(1) &= \chi(S); \end{aligned}$$

$$\begin{aligned} H_S(w^\mu f_1^{\lambda_1} f_2^{\lambda^2} \dots f_r^{\lambda_r}) &= \chi_y(S, E^\mu \otimes F_1^{\lambda_1} \otimes \dots \otimes F_r^{\lambda_r}) \in \mathbb{Z}\{y\}, \\ H_S(1) &= \chi_y(S). \end{aligned}$$

Let s be an indeterminate corresponding to S, then $H_S(w) = \chi_y(S, E)$. Recall the formula (67) for divisor S, which represents $\chi_y(S, E)$ by the χ_y-characteristics of vector bundles over M:

$$\begin{aligned} \chi_y(S, E) &= \sum_{i=0}^{\infty} (-y)^i (\chi_y(M, E \otimes \{S\}^{-i}) - \chi_y(M, E \otimes \{S\}^{-(i+1)})) \\ &= (H_M(w) - H_M(ws^{-1}) - y(H_M(ws^{-1}) - H_M(ws^{-2})) \\ &\quad + y^2(H_M(ws^{-2}) - H_M(ws^{-3})) - y^3(H_M(ws^{-3}) - H_M(ws^{-4})) + \cdots \\ &= H_M(w - ws^{-1} - y(ws^{-1} - ws^{-2}) \\ &\quad + y^2(ws^{-2} - ws^{-3} - y^3(ws^{-3} - ws^{-4}) + \cdots) = H_M\left(w\frac{1 - s^{-1}}{1 - ys}\right), \end{aligned}$$

that is,

$$\chi_y(S, E) = H_M\left(w\frac{1 - s^{-1}}{1 - ys}\right). \tag{68}$$

With this formula, we can generalize to the definition of virtual χ_y-characteristics. Let M be a compact complex manifold of complex dimension n. Let E be a complex holomorphic vector bundle over M. Let $F_1, \dots, F_r$ be r complex analytic line bundles over M. The r-tuple $(F_1, \dots, F_r)$ is called a *virtual submanifold* of M of complex dimension $n-r$. This is a formal definition so we allow $r > n$. To explain the name of virtual submanifold, we may consider the generic case. Suppose each of the line bundle F_i has a global section say s_i, and the zero set S_i of homeomorphic function defined by s_i will be a hypersurface in M. Suppose further that all the S_i intersect transitively, then their intersection $\cap_{i=1,\dots,r} S_i$ gives a complex submanifold of complex dimension $(n-r)$. This may explains the name of "virtual submanifold". Compare with the identity (68), we have

Definition 2.5 (Virtual χ_y-Characteristics). The *virtual χ_y-characteristic* of the virtual submanifold $(F_1, \dots, F_r)$ is defined by

$$\chi_y(F_1, \dots, F_r|, E)_M = H_M\left(w\Pi_{i=1}^r \frac{1 - f_i^{-1}}{1 + yf_i^{-1}}\right).$$

where f_i is the corresponding indeterminate of F_i, for $i = 1, \dots, r$.

When E is the trivial line bundle 1, then write

$$\chi_y(F_1, \dots, F_r) = \chi_y(F_1, \dots, F_r|, 1).$$

This is called the *virtual χ_y-genus* of the virtual submanifold $(F_1, \dots, F_r)$. Furthermore, when $y = 0$, write

$$\chi(F_1, \dots, F_r|, E)_M = \chi_0(F_1, \dots, F_r|, E)_M = h_M(w\Pi_{i=1}^r(1 - f_i^{-1})).$$

This is called the *virtual χ-characteristic* of the restriction to the virtual sub-manifold $(F_1, \ldots, F_r)$. Since $\chi_y(F_1, \ldots, F_r|, E)_M = H_M\left(w\Pi_{i=1}^r \frac{1-f_i^{-1}}{1+yf_i^{-1}}\right)$ is an infinite power series on y with integer coefficients, we can write

$$\chi_y(F_1, \ldots, F_r|, E)_M = \sum_{p=0}^{\infty} \chi^p(F_1, \ldots, F_r|, E)_M y^p.$$

3. Index and its Equivalence to the Euler-Poincaré Characteristics

This section will define index of vector bundles and show its equivalence to the Euler-Poincaré characteristic by the Hodge index theorm. This counts a step in the proof of the Riemann-Roch theorem. The references are [Voisin], [Griffiths] and [Hir].

3.1. Index of Vector Bundles. Suppose V is a finite dimensional real vector space and $Q : V \times V \longrightarrow \mathbb{R}$ is a symmetric bilinear form. If p^+ stands for the number of positive eigenvalues of Q and p^- stands for the number of negative eigenvalues of Q, then the *index* of Q is defined to be $(p^+ - p^-)$. Let M^{4k} be any $4k$-dimensional compact oriented differentiable manifold, there is a natural symmetric bilinear form

$$q : H^{2k}(M, \mathbb{R}) \times H^{2k}(M, \mathbb{R}) \longrightarrow \mathbb{R} \tag{69}$$

defined by $q(x, y) = xy[M]$ where $[M]$ stands for the fundamental cycle of the oriented differentiable manifold M.

Definition 3.1 (Index). The index of the above defined bilinear form q is called the *index of M*, denoted by $\tau(M)$.

3.2. Hodge Decomposition.

Proposition 3.1. *For M being a Riemannian manifold, let $\pi^{(p,q)} : \Omega^*(M) \longrightarrow \Omega^{p,q}(M)$ to be the decomposition of $\Omega^*(M)$ by type. Then $[\Delta_d, \pi^{p,q}] = 0$, where $\Delta_d = dd^* + d^*d$ is the real Lapalace operator.*

Proof. Omitted. $\square$

Let M be a compact Kähler manifold of complex dimension n, by Proposition 3.1 and the identity $\square = \dfrac{\Delta_d}{2}$, we have $[\square, \pi^{(p,q)}] = 0$ when operating on the group of harmonic forms $\mathbb{H}^*(M) = \sum_{p,q} \mathbb{H}^{p,q}(M)$, where $\mathbb{H}^{p,q}(M) = \mathbb{H}^{p,q}(TM)$. Hence we have the well-defined decomposition

$$\mathbb{H}^r(M) = \oplus_{p+q=r} \mathbb{H}^{p,q}(M)$$

for any $0 < r < n$. By the fact that $\square = \frac{\Delta_d}{2}$ is real, we also have $\mathbb{H}^{p,q}(M) = \overline{\mathbb{H}^{q,p}(M)}$. By the Hodge theorem, $H^{p,q}(M) \cong \mathbb{H}^{p,q}(M)$ where $H^{p,q}(M) = H^{p,q}(TM)$, hence the decomposition can be translated into the so called Hodge decomposition.

THEOREM 3.1 (Hodge Decomposition). *Let M be a compact Kähler manifold of complex dimension n, then we have the Hodge decomposition*

$$H^r(M) = \oplus_{p+q=r} H^{p,q}(M)$$

for all $0 < r < n$ *and* $H^{p,q}(M) = \overline{H^{q,p}(M)}$.

3.3. Lefschetz Decomposition. This subsection is devoted to the hard Lefschetz theorem and its corollary, the Lefschetz decomposition theorem. The references are [Griffiths] and [Voisin]. The Lefschetz decomposition, together with the Hodge decomposition, are used to prove the Hodge index theorem in the next subsection.

As a motivation, let us consider how to get a flag in a Euclidean space $\mathbb{R}^n$ and hence get a decomposition of $\mathbb{R}^n$ by the flag. We can determine a flag by giving its generators at each stage: Note that any k-plane P^k with $0 \leq k \leq n$ through the origin in the Euclidean space $\mathbb{R}^n$ can be described by its basis, i.e. k linearly independent vectors in $\mathbb{R}^n$ or the duals of the k vectors, namely k covectors, which are k (n)-forms. Or equivalently, n (k)-forms linear independently. Wedging each of the n (k)-forms with a 1-form ω_k, which is not linearly dependent on any of the original k-forms, we get n $(k+1)$-forms which represent a $(k+1)$-plane P^{k+1} in $\mathbb{R}^n$ which contains P^k as a subspace. Choose another 1-form ω_{k+1}, which is linearly independent of the previous n-$(k+1)$ covectors and wedge ω_{k+1} to each of the covectors. We will get a $(k+2)$-plane P^{k+2} containing P^{k+1} as a subspace. Repreat the process by choosing a good 1-form at each step, we end up with a flag $(\cdots \subset P^k \subset P^{k+1} \subset \cdots \subset P^n = \mathbb{R}^n)$. This flag gives a decomposition of the vector space $\mathbb{R}^n$. Note that it is possible to choose a 1-form which is good for all the stages.

Instead of considering $\mathbb{R}^n$, the same strategy applies to decompose $\mathbb{C}^n$. More generally, since a Kähler manifold has pointwisely special coordinates made the Kähler metric trivial, hence we may regard a Kähler manifold M of dimension n be locally a Euclidean space $\mathbb{C}^n$ with flat metric. The decomposition strategy can be applied pointwisely over tangent space of points in M. Once we can make the decomposition consistently with respect to different points in the Kähler manifolds, we will have a global decomposition of tangent bundle of M. Represent the generators in differential forms this decomposition will be a decomposition of differential forms or further cohomology groups. There is a natural good $(1,1)$-form, namely the Kähler form, to realize such decomposition. This is known as the Lefschetz decomposition. We will first give some Kähler identities as preparation, then give the hard Lefschetz theorem which can be viewed as a particular isomorphism between $H^k(M)$ and $H^{2n-k}(M)$ by wedged with proper times of Kähler form, then use this isomorphism to get the Lefschetz decomposition.

DEFINITION 3.2 (Lefschetz Operator). Let M be a Kähler manifold of complex dimension n with Kähler form ω, which is a $(1,1)$-form. The *Lefschetz operator* $L : \Omega^{p,q}(M) \longrightarrow \Omega^{p+1,q+1}(M)$ is defined by $\wedge\omega$.

Since ω is real, L maps $\bar{\partial}$-closed form to $\bar{\partial}$-closed form and $\bar{\partial}$-exact form to $\bar{\partial}$-exact form. Hence it induces naturally an operator also denote as L, $L : H^{p,q}(M) \longrightarrow H^{p+1,q+1}(M)$. This is an alternative definition of the Lefschetz operator. Recall from 1.2.2, there is an L^2-metric $(\cdot,\cdot)_{L^2}$ on $\Omega^{p,q}(M)$ induced from the Kähler metric ω. Let $\Lambda : \Omega^{p,q}(M) \longrightarrow \Omega^{p-1,q-1}(M)$, or equivalently, $\Lambda : H^{p,q}(M,\mathbb{R}) \longrightarrow H^{p-1,q-1}(M,\mathbb{R})$ be the adjoint L^* of L under the L^2-metric on $\Omega^{p,q}(M)$. That is $L^* = - * L*$ where $*$ is defined in (59). Before giving the Kähler identities concerning the properties of the Lefschetz operator and its adjoint, we will give some preparation on linear algebra.

3.3.1. *Algebraic Preliminary.* Let u be a tangent vector of some differentiable manifold M and $\xi \in A^k(M,\mathbb{R})$, then the *interior product* $int(u)(\xi)$ is defined to be the differential $(k-1)$-form such that

$$int(u)(\xi)(v_1,\ldots,v_{k-1}) = \xi(u,v_1,\ldots,v_{k-1})$$

for any $(k-1)$ tangent vectors $v_1,\ldots,v_{k-1}$ of M.

Proposition 3.2. *On $\mathbb{R}^n$, let $\{x_1,\ldots,x_n\}$ be the local coordinates and the metric is given by $\sum_i dx^i \otimes dx^i$. Then $*^{-1}(dx^i \wedge *dx^I) = (-1)^{k+1} int\left(\frac{\partial}{\partial x_i}\right)(dx^I)$ where $I \subset \{1,\ldots,n\}$ with $|I| = k$ and $*$ is defined in (58).*

Proof. Direct computation. ☐

Proposition 3.3. *On $\mathbb{C}^n$, let $\{z_1,\ldots,z_n\}$ be the local coordinates and the metric is given by $\sum_i dz^i \wedge d\bar{z}^i$. Then $*(dz^i\wedge)* = -2int\left(\frac{\partial}{\partial \bar{z}_i}\right)$ and $*(d\bar{z}^i\wedge)* = -2int\left(\frac{\partial}{\partial z_i}\right)$ for $i = 1,\ldots,n$.*

Proof. We may assume that $z_i = x_i + \sqrt{-1}y_i$, $dz^i = dx^i + \sqrt{-1}dy^i$ and $\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right)$ and $\frac{\partial}{\partial \bar{z}_i} = \frac{1}{2}\left(\frac{\partial}{\partial \bar{x}_i} + \sqrt{-1}\frac{\partial}{\partial y_i}\right)$. To calculate $*(dz^i\wedge)*$, it is enough to calculate $*^{-1}(dx^i\wedge)*$ and $*^{-1}(dy^i\wedge)*$. By Proposition 3.2,

$$*^{-1}(dx^i\wedge) * (dx^I) = (-1)^{k+1} int\left(\frac{\partial}{\partial x_i}\right)(dx^I);$$
$$*^{-1}(dy_i\wedge) * (dy^J) = (-1)^{k+1} int\left(\frac{\partial}{\partial y_i}\right)(dy^J)$$

where $I, J \subset \{1, dots, n\}$ and $|I| = |J| = k$. Recall that $*^2 = (-1)^k$ and hence $* = (-1)^k *^{-1}$. Hence

$$\begin{aligned}
& *(dz^i \wedge) * (dx^I + dy^J) \\
= & *((dx^i + \sqrt{-1}dy^i)\wedge) * (dx^I + dy^J) \\
= & (-1)^k *^{-1} ((dx^i + \sqrt{-1}dy^i)\wedge) * (dx^I + dy^J) \\
= & (-1)^k *^{-1} (dx^i \wedge) * (dx^I) + \sqrt{-1}(-1)^k *^{-1} (dy^i \wedge) * (dy^J) \\
= & (-1)^k(-1)^{k+1} int \left(\frac{\partial}{\partial x_i}\right)(dx^I) + \sqrt{-1}(-1)^k(-1)^{k+1} int \left(\frac{\partial}{\partial y_i}\right)(dy^J) \\
= & int \left(-\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial y_i}\right)(dx^I + dy^J) = -2int\left(\frac{\partial}{\partial \bar{z}_i}\right)(dx^I + dy^J)
\end{aligned}$$

and hence $*(dz^i\wedge)* = -2int\left(\frac{\partial}{\partial \bar{z}_i}\right)$ for $i = 1, \ldots, n$. $*(d\bar{z}^i\wedge)* = -2int\left(\frac{\partial}{\partial z_i}\right)$ can be proven similarly. □

3.3.2. *Kähler Identities.*

PROPOSITION 3.4. *Over a compact Kähler manifold M, $[\Lambda, \bar{\partial}] = -i\partial^*$, $[\Lambda, \partial] = i\bar{\partial}^*$. Or equivalently, $[L, \bar{\partial}^*] = -i\partial$, $[L, \partial^*] = i\bar{\partial}]$.*

PROOF. Direct computation. □

PROPOSITION 3.5. *The Laplacian commute with L, i.e., $[\Delta, L] = 0$.*

PROOF.

$$\begin{aligned}
[\Delta, L] &= [\partial\partial^*, L] + [\partial^*\partial, L] \\
&= \partial[\partial^*, L] + [\partial^*, L]\partial \quad \text{by } [L, \partial] = 0 \\
&= \partial(-i\bar{\partial}) + (-i\bar{\partial})\partial = 0
\end{aligned}$$

□

3.3.3. *Hard Lefschetz Theorem.*

LEMMA 3.6. *Let M be a Kähler manifold of complex dimension n, then we have pointwisely, $[L, \Lambda] = (k - n)Id$ on $\Omega^k(M)$ where $0 \leq k \leq n$.*

PROOF. It suffices to show the formula on $\mathbb{C}^n$ with flat metric since we can choose special coordinate for the Kähler manifold pointwisely. Define A_i to be the operator $\wedge\frac{\sqrt{-1}}{2}\sum_i dz^i \wedge d\bar{z}^i$ for $i = 1, \ldots, n$. Hence

$$L = \sum_i A_i; \quad \Lambda = \sum_i *^{-1}A_i * .$$

Hence, $[L,\Lambda]=\sum_{i,j}[A_i, *^{-1}A_j*]$. By using Proposition 3.3, we have

$$\begin{aligned}
^{-1}A_i &= *^{-1}(\wedge\frac{\sqrt{-1}}{2}dz^i\wedge d\bar{z}^i)*\\
&= \frac{\sqrt{-1}}{2}\left(*^{-1}(dz^i\wedge)*\right)\wedge\left(*^{-1}(d\bar{z}^i\wedge)*\right)\\
&= \frac{\sqrt{-1}}{2}\cdot(-2)int\left(\frac{\partial}{\partial z_i}\right)\wedge(-2)int\left(\frac{\partial}{\partial\bar{z}_i}\right)=2\sqrt{-1}int\left(\frac{\partial}{\partial z_i}\wedge\frac{\partial}{\partial\bar{z}_i}\right).
\end{aligned}$$

Hence $A_i *^{-1} A_j* = *^{-1}A_j * A_i$ for $0\le i\ne j\le n$. Indeed,

$$A_i*^{-1}A_j*=A_i2\sqrt{-1}int\left(\frac{\partial}{\partial z_j}\wedge\frac{\partial}{\partial\bar{z}_j}\right)=2\sqrt{-1}int\left(\frac{\partial}{\partial z_j}\wedge\frac{\partial}{\partial\bar{z}_j}\right)A_i=*^{-1}A_j*A_i.$$

Therefore,

$$\begin{aligned}
[\Lambda,L] &= \sum_i[A_i,*^{-1}A_i*]\\
&= \sum_i\left[\frac{\sqrt{-1}}{2}dz^i\wedge d\bar{z}^i, 2\sqrt{-1}int\left(\frac{\partial}{\partial z_i}\wedge\frac{\partial}{\partial\bar{z}_i}\right)\right]\\
&= \sum_i\left[dz^i\wedge d\bar{z}^i, int\left(\frac{\partial}{\partial z_i}\wedge\frac{\partial}{\partial\bar{z}_i}\right)\right].
\end{aligned}$$

Before going on, let us introduce some notations to simplify the expressions. For $Q\subset\{1,\dots,n\}$. Set $\omega^Q=\wedge_{q\in Q}dz^q\wedge d\bar{z}^q$. Any form η of degree k can be written as a linear combination of the forms $\{\eta_{A,B,Q}=dz^A\wedge d\bar{z}^B\wedge\omega^Q\}$ where $A,B,Q\subset\{1,\dots,n\}$ are disjoint subset and further with $|A|+|B|+2|Q|=k$. If A,B and Q are fixed, then let $J=\{1,\dots,n\}-(A\cup B\cup Q)$. Observe the following algebraic relations:

$$\begin{aligned}
dz^i\wedge d\bar{z}^i\wedge\eta_{A,B,Q} &= 0 \quad \text{if } i\in A\cup B\cup Q\\
int\left(\frac{\partial}{\partial z_i}\wedge\frac{\partial}{\partial\bar{z}_i}\right)(\eta_{A,B,Q}) &= 0 \quad \text{if } i\in\{1,\dots,n\}-Q\\
(dz^i\wedge d\bar{z}^i)\circ int\left(\frac{\partial}{\partial z_i}\wedge\frac{\partial}{\partial\bar{z}_i}\right)(\eta_{A,B,Q}) &= \eta_{A,B,Q} \quad \text{if } i\in Q\\
int\left(\frac{\partial}{\partial z_i}\wedge\frac{\partial}{\partial\bar{z}_i}\right)\circ(dz^i\wedge d\bar{z}^i)(\eta_{A,B,Q}) &= \eta_{A,B,Q} \quad \text{if } i\in J.
\end{aligned}$$

Hence we have

$$\begin{aligned}
[L,\Lambda](\eta_{A,B,Q}) &= \sum_i[dz^i\wedge d\bar{z}^i, int\left(\frac{\partial}{\partial z_i}\wedge\frac{\partial}{\partial\bar{z}_i}\right)](\eta_{A,B,Q})\\
&= \sum_i dz^i\wedge d\bar{z}^i\cdot int\left(\frac{\partial}{\partial z_i}\wedge\frac{\partial}{\partial\bar{z}_i}\right)(\eta_{A,B,Q})\\
&\quad -int\left(\frac{\partial}{\partial z_i}\wedge\frac{\partial}{\partial\bar{z}_i}\right)\cdot dz^i\wedge d\bar{z}^i(\eta_{A,B,Q})=(|Q|-|J|)\eta_{A,B,Q}.
\end{aligned}$$

However, $|J| = n = |A| - |B| - |Q|$ together with $|A| + |B| + 2|Q| = k$ implies that $|Q| - |J| = k - n$. Therefore, $[L, \Lambda](\eta_{A,B,Q}) = (k-n)(\eta_{A,B,Q})$ for any k form $\eta_{A,B,Q}$ and hence the result. □

LEMMA 3.7. *Let M be a Kähler manifold of complex dimension n. Then the morphism of vetor bundles $L^{n-k} : \Omega^k(M) \longrightarrow \Omega^{2n-k}(M)$ where $0 \leq k \leq n$ is an isomorphism.*

PROOF. Since the preimage and the image space have the same dimension, it suffices to show that the map is injective. For a fixed k and an $r \leq n-k$, we will show the mapping $L^r : \Omega^k(M) \longrightarrow \Omega^{k+2r}(M)$ is injective. If this is shown then in particular if $r - n - k$, this will be the statement we want to show. We will prove the statement by induction on both r and the degree k. We assume that it is injective in the case $< r$ or $< k$. Let $\xi \in \Omega^k(M)$ such that $L^r(\xi) = 0$, we want to show that $\xi = 0$. Lemma 3.6 gives $[L, \Lambda] = (k-n)Id$, hence $[L^r, \Lambda] = r(k-n)L^{r-1}$. Indeed,

$$\begin{aligned}
[L^r, \Lambda] &= [LL^{r-1}, \Lambda] \\
&= LL^{r-1}\Lambda - L\Lambda L^{r-1} + L\Lambda L^{r-1} - \Lambda LL^{r-1} \\
&= L[L^{r-1}, \Lambda] + [L, \Lambda]L^{r-1} \\
&= L(L[L^{r-2}, \Lambda] + [L, \Lambda]L^{r-2}) + [L, \Lambda]L^{r-1} \\
&= L^2[L^{r-2}, \Lambda] + L(k-n)IdL^{r-2} + (k-n)IdL^{r-1} \\
&= L^2[L^{r-2}, \Lambda] + 2(k-n)L^{r-1} \\
&\vdots \\
&= L^{r-1}[L, \Lambda] + (r-1)(k-n)L^{r-1} \\
&= L^{r-1}(k-n)Id + (r-1)(k-n)L^{r-1} = r(k-n)L^{r-1}.
\end{aligned}$$

Operate on our ξ,

$$\begin{aligned}
[L^r, \Lambda](\xi) &= r(k-n)L^{r-1}(\xi) \\
L^r\Lambda(\xi) - \Lambda L^r(\xi) &= r(k-n)L^{r-1}(\xi) \\
L^r\Lambda(\xi) &= r(k-n)L^{r-1}(\xi) \\
L\Lambda(\xi) &= r(k-n)(\xi).
\end{aligned}$$

By the induction assumption with respect to r , L^{r-1} is injective on Ω^{k-1}. Now let $\eta = -\dfrac{1}{r(k-n)}\Lambda(\xi) \in \Omega^{n-r-2}$, hence $L(\eta) = \xi$. This implies that $L^{r+1}\eta = 0$. By the induction assumptions on degree of ξ, we have L^{r+1} when operating on a $(k-2)$-form is injective, hence $\eta = 0$ and then $\xi = 0$. Therefore, injectivity is shown and hence the theorem. □

THEOREM 3.2 (Hard Lefschetz Theorem). *Let M be a compact Kähler manifold of complex dimension n, then for every $0 \leq k \leq n$, $L^{n-k} : H^k(M) \longrightarrow H^{2n-k}(M)$ is an isomorphism.*

PROOF. By the Poincaré duality, $H^k(M)$ and $H^{2n-k}(M)$ has the same dimension, hence it suffices to prove injectivity of L^{n-k}. By the Hodge theorem, cohomology classes can be represented by in particular harmonic forms, hence Lemma 3.7 stated for forms is true when apply to harmonic forms and hence the theorem. □

3.3.4. *Lefschetz Decomposition.* Let M be a compact Kähler manifold of complex dimension n. An element $\alpha \in \Omega^k(M), k \leq n$ is called *primitive*, if $L^{n-k+1}\alpha = 0$.

PROPOSITION 3.8. *$\alpha \in \Omega^k(M)$ being primitive is equivalent to (i) $\Lambda\alpha = 0$ or (ii) $L^{n-k+1}\alpha = 0$ for $k \leq n$.*

PROOF. If $\alpha = 0$ then the statement trivially holds. Suppose $\alpha \neq 0$. (i) If $L^{n-k}\alpha \neq 0$, then $L^{n-k+1}\alpha = 0 \iff (L^{n-k+1}\alpha, \alpha)_{L^2} = 0(L^{n-k}\alpha, \Lambda\alpha)_{L^2} = 0 \iff \Lambda\alpha = 0$. If $L^{n-k}\alpha = 0$, then if $L^{n-k-1}\alpha \neq 0$ the same proof adopts when replacing k by $k-1$ if not then go on the process by induction and get the proof.

(ii) The "if part" is obvious. For "the only if part", by using the equivalent statement $\Lambda\alpha = 0$, we have $\Lambda\alpha = 0 \Longrightarrow (\alpha, \Lambda\alpha)_{L^2} = 0 \Longrightarrow (\alpha, \Lambda^{n-k+1}\alpha)_{L^2} = 0 \Longrightarrow (L^{n-k+1}\alpha, \alpha)_{L^2} = 0 \Longrightarrow L^{n-k+1}\alpha = 0$ for $0 \leq k \leq n$. □

THEOREM 3.3 (Lefschetz Decomposition Theorem). *Let M be a compact Kähler manifold and L the corresponding Lefschetz operator. Then every cohomology class $\alpha \in H^k(M)$ admits a unique decomposition $\alpha = \sum_r L^r\alpha_r$, where the α_r are of degree $(k-2r) \leq \min(k, 2n-k)$ and are primitive in the sense that $L^{n-k+2r+1}\alpha_r = 0$ in $H^{2n-k+2r+2}(M, \mathbb{R})$. That is we have the Lefschetz decomposition:*

$$H^k(M, \mathbb{R}) = \oplus_{2r \leq k} L^r H^{k-2r}(X, \mathbb{R})_{prim}$$

where V_{prim} stands for the set of all the primitive elements in V.

PROOF. Existence: Without loss of generality, we may assume that $k = deg\alpha \leq n$, consider image of α, $L^{n-k+1}\alpha \in \Omega^{2n-k+2}$ under the mapping $L^{n-k+1} : \Omega^k \longrightarrow \Omega^{2n-k+2}$. By the hard Lefschetz theorem, the isomorphisms $L^{n-k+2} : \Omega^{k-2} \longrightarrow \Omega^{2n-k+2}$ gives a preimage $\beta \in \Omega^{k-2}$ such that $L^{n-k+2}\beta = L^{n-k+1}\alpha$. Hence $\alpha = L\beta + \alpha_0$ where $\alpha_0 = \alpha - L\beta$ is primitive. Indeed

$$\begin{aligned} 0 &= L^{n-k+2}\beta - L^{n-k+1}\alpha \\ &= L^{n-k+1}(L\beta - \alpha) = L^{n-k+1}(\alpha_0). \end{aligned}$$

We can similarly find the decomposition of β and by induction we get the decomposition $\alpha = \sum_r L^r\alpha_r$ fro α_r of degree $(k-2r)$, which is primitive.

Uniqueness: Also assume that $k \leq n$. Suppose we for the same α we have two decomposition

$$\sum_r L^r\alpha_r'; \quad \sum_r L^r\alpha_r''.$$

Write $\alpha_r = \alpha_r' - \alpha_r''$, then $\sum_r L^r\alpha_r = 0$ with α_r being primitive and of degree $k-2r$. Note that now $0 \in \Omega^k(M, \mathbb{R})$ and each of the component is now an integer. Denote the integers $L^r\alpha_r$ by $b_r, r = 1, \ldots, n$. Then if $min\{b_0, \ldots, b_n\} \neq$

0, then we can write $L\left(\sum_r L^{r-1}\alpha_r\right) = 0$. By the hard Lefschetz theorem, L is injective on $\Omega^{n-1}(M)$, hence

$$\left(\sum_r L^{r-1}\alpha_r\right) = 0.$$

By induction, $\sum_r \alpha_r = 0$, we can conclude that $\alpha_r = 0$ for each r by comparing degrees of α_r's. If $min\{b_0, \dots, b_n\} = 0$, $L^{n-k+1}\sum_{r>0} L^r\alpha_r = -L^{n-k+1}L^0\alpha_0 = 0$ implies that $L^{n-k+2}\sum_{r>0} L^{r-1}\alpha_r = 0$ and the hard Lefschetz theorem implies that $\sum_{r>0} L^{r-1}\alpha_r = 0$ by induction on the degree we have $\alpha_r = 0$ for $r > 0$ and thus $\alpha_0 = 0$. Uniqueness is proved. □

3.4. Hodge Index Theorem. Let M be a Hermitian manifold of complex dimension n and ω be the corresponding Kähler form. Let L be the Lefschetz operator with respect to the Kähler form. We state the proposition whose proof can be found in [Voisin] p.151.

PROPOSITION 3.9. *Let α be a primitive element in $\Omega^{p,q}_x(M) \subset \Omega^k_x(M)$ where $p + q = k$. Then, $*\alpha = (-1)^{\frac{k(k+1)}{2}}\sqrt{-1}^{p-q}\dfrac{L^{n-k}}{(n-k)!}\alpha$ where $*$ is as defined in (59).*

Let M be a compact Kähler manifold of complex dimension n and let ω be its Kähler form. Consider the pairing

$$\tau : \Omega^k(M) \times \Omega^{2n-k}(M) \longrightarrow \mathbb{C}$$

given by $\tau(\alpha, \beta) = \int_M \alpha \wedge \beta$ for $\alpha \in \Omega^k(M), \beta \in \Omega^{2n-k}(M)$ and $0 \leq k \leq n$. This can be canonically generalized to the pairing

$$\tilde{\tau} : H^k(M, \mathbb{C}) \times H^{2n-k}(M, \mathbb{C}) \longrightarrow \mathbb{C}$$

with no ambiguity by using the Stokes' theorem. For k being even, define the *intersection form*

$$H_k : H^k(M, \mathbb{C}) \times H^k(M, \mathbb{C}) \longrightarrow \mathbb{C}$$

by $H_k(\alpha, \beta) = \tilde{\tau}(L^{n-k}\alpha, \bar{\beta})$ for $\alpha, \beta \in H^k(M, \mathbb{C})$ where $0 \leq k \leq n$. That is $H_k(\alpha, \beta) = \int_M L^{n-k}\alpha \wedge \bar{\beta} = \int_M \wedge^{2n-2k}\omega \wedge \alpha \wedge \bar{\beta}$. It can be checked that this H_k is a symmetric bilinear form. Note also that when the complex dimension n of M is even and restrict on the coefficient ring $\mathbb{R}$, then $H_n = q$ where q is the symmetric bilinear form q defined in (69) at the beginning of the subsection 3.1. We will show the index of q is equal to the Euler characteristics in the Hodge index theorem.

LEMMA 3.10. *The Lefschetz decomposition*

$$H^k(M, \mathbb{C}) = \oplus_{2r \leq k} L^r H^{k-2r}(X, \mathbb{C})_{prim}$$

is an orthogonal decomposition with respect to H_k, where k is even and $\leq n$.

PROOF. For any $0 \leq r < s \leq \frac{k}{2}$, if $\alpha = L^r\alpha'$ and $\beta = L^s\beta'$ for α', β' being primitive elements. Then

$$H_k(\alpha, \beta) = \tilde{\tau}(L^{n-k}\alpha, \bar{\beta}) = \int_M L^{n-k}\alpha \wedge \bar{\beta}.$$

Since

$$L^{n-k}\alpha \wedge \bar{\beta} = L^{n-k}(L^r\alpha' \wedge L^s\bar{\beta}') = (L^{n-k+r+s}\alpha) \wedge \bar{\beta}' = 0$$

by the assumption that α' is primitive. Hence $H_k(\alpha, \beta) = 0$ and hence the Lefschetz decomposition is orthogonal with respect to H_k. □

LEMMA 3.11. *The Hodge decomposition $H^k(M, \mathbb{C}) = \oplus_{p+q=k} H^{p,q}(M)$ forms an orthogonal direct sum with respect to H_k, where k is even and $k \leq n$. Moreover, the form $(-1)^{\frac{k(k-1)}{2}} i^{p-q} H_k$ is positive definite on the complex subspaces*

$$H^{p,q}_{prim} = H^k(M, \mathbb{C})_{prim} \cap H^{p,q}(M).$$

PROOF. For any pair $(p, q), (p', q')$ such that $p + q = p' + q' = k$ and $p \neq p'$ and for any $\alpha^{p,q}, \beta^{p',q'} \in H^k(M, \mathbb{C})$,

$$H_k(\alpha, \beta) = \tilde{\tau}(L^{n-k}\alpha^{p,q}, \bar{\beta}^{p',q'}) = \int_M L^{n-k}\alpha^{p,q} \wedge \bar{\beta}^{p',q'} = 0,$$

since the integrand $L^{n-k}\alpha^{p,q} \wedge \bar{\beta}^{p',q'}$ is of type $(2n-2k+p+q', 2n-2k+q+p') = (2n-2k+p+q', 2n-p-q')$ which is by no means equal to (n, n).

For the second part, for any primitive form $\alpha^{p,q} \in H^{p,q}_{prim}$, α is primitive if and only if its representative in $Z^{p,q}(M)$ is primitive. Hence we are safe to apply the identity given by Proposition 3.9 on $\bar{\alpha}$, that is, $*\alpha = (-1)^{\frac{k(k+1)}{2}} \sqrt{-1}^{p-q} \frac{L^{n-k}}{(n-k)!}\alpha$. Or equivalently,

$$L^{n-k}\alpha = (-1)^{\frac{k(k+1)}{2}} \sqrt{-1}^{q-p} (n-k)! * \alpha.$$

Compute

$$\begin{aligned} H_k(\alpha) &= \int_M L^{n-k}\alpha \wedge \bar{\alpha} \\ &= (-1)^{\frac{k(k+1)}{2}} \sqrt{-1}^{q-p} (n-k)! \int_M *\alpha \wedge \bar{\alpha} \\ &= \sqrt{-1}^{q-p} (-1)^{\frac{k(k+1)}{2}} (n-k)! \int_M \bar{\alpha} \wedge *\alpha \\ &= \sqrt{-1}^{q-p} (-1)^{\frac{k(k+1)}{2}} (n-k)! ||\bar{\alpha}||_{L^2}. \end{aligned}$$

Hence

$$\begin{aligned} (-1)^{\frac{k(k-1)}{2}} \sqrt{-1}^{p-q} H_k(\alpha) &= (-1)^{\frac{k(k-1)}{2}} \sqrt{-1}^{p-q} \sqrt{-1}^{q-p} (-1)^{\frac{k(k+1)}{2}} (n-k)! ||\bar{\alpha}||_{L^2} \\ &= (n-k)! ||\bar{\alpha}||_{L^2} > 0 \end{aligned}$$

whenever $\alpha \neq 0$. □

THEOREM 3.4 (Hodge Index Theorem). *Let M^n be a n-complex dimensional compact Kähler manifold and n is an even number. Then the index of M^n $q : H^n(M,\mathbb{R}) \times H^n(M,\mathbb{R}) \longrightarrow \mathbb{R}$ is equal to the Euler characteristic $\chi(M) = \sum_{a,b}(-1)^a h^{a,b}(M)$ where $h^{a,b} = dim(H^{a,b}(M))$.*

PROOF. Note that the index of q is the same as the index of the symmetric form

$$H_n : H^n(M,\mathbb{C}) \times H^n(M,\mathbb{C}) \longrightarrow \mathbb{C}.$$

By Lemma 3.10, the Lefschetz decomposition

$$H^n(M,\mathbb{C}) = \oplus_{a+b=n-2r} L^r H^{a,b}_{prim},$$

is orthogonal with respect to H_n. Further by Lemma 3.11, the Hodge decomposition

$$H^n(M,\mathbb{C}) = \oplus_{p+q=n} H^{p,q}(X,\mathbb{C})$$

is also orthogonal with respect to H_n. Hence we may combine the two decompositions by

$$H^n(M,\mathbb{C}) = \sum_{\substack{a+b=0 \\ (a+b=0 mod 2)}}^{n} L^{\frac{n-(a+b)}{2}} H^{a,b}_{prim}(M) \tag{70}$$

where $H^{a,b}_{prim}(M) = H^{a+b}_{prim} \cap H^{a,b}(M)$. Note that this decomposition is also orthogonal with respect to H_n. Hence the index of H_n on $H^n(M,\mathbb{C})$ is equal to the sum of the indices of H_n on every components in (70). By the fact that

$$H_{s-2r}(\alpha,\beta) = H_s(L^r\alpha, L^r\beta)$$

for $L^r\alpha \in H^s(M,\mathbb{C})_{prim}$, the index of H_n operating on the space $L^{\frac{n-(a+b)}{2}} H^{a,b}_{prim}(M)$ agrees with the index of $H_{n-(n-(a+b))} = H_{a+b}$ operating on the space $H^{a+b}_{prim}(M,\mathbb{C})$. Hence the index of H_n can be written in the sum

$$\tau(H_n) = \sum_{\substack{a+b=0 \\ (a+b=0 mod 2)}}^{n} \text{index of } H_{a+b} \text{ on } H^{a+b}_{prim}.$$

By Lemma 3.11, $(-1)^{\frac{(a+b)(a+b-1)}{2}} i^{a-b} H_{a+b}$ operate positive definite on the space $H^{a,b}_{prim}$, hence

$$\text{index of } H_{a+b} = (-1)^{\frac{(a+b)(a+b-1)}{2}} i^{a-b} h^{a,b}_{prim}$$

where $h^{a,b}_{prim} = dim H^{a,b}_{prim}(M,\mathbb{C})$. Hence,

$$\tau(H_n) = \sum_{\substack{a+b=0 \\ (a+b=0 mod 2)}}^{n} (-1)^{\frac{(a+b)(a+b-1)}{2}} i^{a-b} h^{a,b}_{prim} = \sum_{\substack{a+b=0 \\ (a+b=0 mod 2)}}^{n} (-1)^a h^{a,b}_{prim}.$$

From Proposition 3.8, we immediately have $h^{a,b}_{prim} = h^{a,b} - h^{a-1,b-1}$, where $h^{a,b} = dimH^{a,b}(M, \mathbb{C})$. Therefore,

$$\begin{aligned}
\tau(H_n) &= \sum_{\substack{a+b=0 \\ (a+b=0mod2)}}^{n} (-1)^a h^{a,b}_{prim} \\
&= \sum_{\substack{a+b=0 \\ (a+b=0mod2)}}^{n} (-1)^a (h^{a,b} - h^{a-1,b-1}) \\
&= \sum_{\substack{a+b=0 \\ (a+b=0mod2)}}^{n} (-1)^a h^{a,b} - \sum_{\substack{a+b=0 \\ (a+b=0mod2)}}^{n} (-1)^a h^{a-1,b-1}) \\
&= \sum_{\substack{a+b=0 \\ (a+b=0mod2)}}^{n} (-1)^a h^{a,b} + \sum_{\substack{a'+b'=-2 \\ (a'+b'=0mod2)}}^{n-2} (-1)^{a'} h^{a',b'}) \\
&= 2 \sum_{\substack{a+b=0 \\ (a+b=0mod2)}}^{n-2} (-1)^a (h^{a,b} + \sum_{a+b=n} (-1)^a h^{a,b} \\
&= \sum_{\substack{a+b=0 \\ (a+b=0mod2)}}^{n-2} (-1)^a h^{a,b} + \sum_{\substack{a+b=0 \\ (a+b=0mod2)}}^{n-2} (-1)^a h^{a,b} + \sum_{a+b=n} (-1)^a h^{a,b} \\
&= \sum_{\substack{a+b=0 \\ (a+b=0mod2)}}^{n-2} (-1)^a h^{a,b} + \sum_{\substack{a'+b'=n+2 \\ (a'+b'=0mod2)}}^{2n} (-1)^{n-a'} h^{a',b'} \\
&\quad + \sum_{a+b=n} (-1)^a h^{a,b} \quad (a' = n-a; b' = n-b \text{ in the second term}) \\
&= \sum_{\substack{a+b=0 \\ (a+b=0mod2)}}^{2n} (-1)^a h^{a,b}
\end{aligned}$$

Note the Poincaré duality is used in the second last line and the condition that n is even is used in the last line. The Hodge index theorem is proven. □

CHAPTER 5

Todd Theory

We have seen the representation of the Euler-characteristic of a compact differentiable manifold by the characteristic classes. With the proper generalization of both the characteristic classes of manifolds to the characteristic classes of vector bundles and the Euler characteristics of manifolds to the Euler-Poincaré characteristics of vector bundles, it is believed that the Euler-Poincaré characteristic can be represented by characteristic classes. It is remarkable that Todd in 1937 represented the Euler-Poincaré characteristics of vector bundles by the so called *Todd classes* by using Todd sequences, although his proof was incomplete. In 1953, Nakono gave the proof that the Todd classes are equivalent to Chern classes. A specific polynomial relation between the Euler-Poincaré characteristics and the Chern classes is thus available. Instead of adopting Todd's incomplete proof, Hirzebruch proved that Todd sequence in terms of Chern classes is equal to the Euler-Poincaré characteristics in [Hir]. This is known as the Riemann-Roch theorem of Hirzebruch. We will study Todd theory in this chapter.

1. Todd Theory

In this section, we will first give basic definitions of *multiplicative sequences*, their *characteristic polynomial* Q, and the useful equivalent expression

$$\sum_j T_j(c_1, \dots, c_j) = \Pi_{i=1}^n Q(\gamma_i)$$

where $\sum_{j=0}^n c_j x^j = \Pi_{i=1}^n (1 + \gamma_i x)$. We will then define the *total Todd class* which is a series of Todd polynomials with the Chern classes as indeterminates, and define the corresponding *Todd genus*, which is the pairing between the total Todd class and the fundamental homology class. After this, we will give the *generalized Todd classes* T_y by adding one more indeterminates y and also the *generalized Todd genus* by pairing T_y with the fundamental homology class. With the generalized Todd genus, we can define the Todd genus for a codimension 1 submanifold and further for codimension r virtual submanifolds, that is the *virtual generalized Todd-genus.* After defining the *Chern character*, another series of polynomials in terms of Chern classes, we can as an analogy of the above process get all kinds of classes and characteristics of vector bundles instead of the tangent bundles. That is, the *total T-classes*, the *T-characteristics,* the *generalized T-characteristics* and the *virtual generalized T-characteristics.*

1.1. Multiplicative Sequences and Todd Polynomials. Let B be a commutative ring with unity. Let $p_1, p_2, \ldots$ be indeterminates such that p_i is of degree i. Define a polynomial ring $\mathcal{B} = B[p_1, p_2, \ldots]$. Define the weight of the product $p_{j_1} p_{j_2} \ldots p_{j_r}$ by $(j_1 + j_2 + \cdots + j_r)$ and denote $\mathcal{B}_k$ the set of all homogeneous polynomials in $\mathcal{B}$ with weight k. Therefore, $\mathcal{B}_k$ is an additive group with the addition of polynomials. It can be checked that the polynomial ring $\mathcal{B}$ satisfies the conditions (i)$\mathcal{B} = \oplus_{k \geq 0} \mathcal{B}_k$; (ii) $\mathcal{B}_r \mathcal{B}_s \subset \mathcal{B}_{r+s}$. Hence $\mathcal{B}$ is graded.

Definition 1.1 (Multiplicative sequence). Let $\{K_j\}$ be a sequence of polynomials in the indeterminates p_i with $K_0 = 1$ and $K_j \in \mathcal{B}_j, j = 0, 1, 2, \ldots$. The sequence $\{K_j\}$ is called a *multiplicative sequence* (or *m-sequence*) if whenever we have an identity like

$$1 + p_1 z + p_2 z^2 + \cdots = (1 + p_1' z + p_2' z^2 + \cdots)(1 + p_1'' z + p_2'' z^2 + \cdots),$$

we also have

$$\sum_{i=0}^{\infty} K_j(p_1, p_2, \ldots, p_i) z^i = \sum_{j=0}^{\infty} K_i(p_1', p_2', \ldots, p_j) z^j \sum_{k=0}^{\infty} K_k(p_1'', p_2'', \ldots, p_k'') z^k.$$

where z is an indeterminate and z^i is the i-th power of z and therefore of degree i. In abbreviation, we write

$$\sum_{i=0}^{\infty} K_i(p_1, \ldots, p_i) z^i = K\left(\sum_{i=0}^{\infty} p_i z^i\right).$$

Note that we use the same notation when p_i is replaced by particular values. In particular, if we choose $p_0 = p_1 = 1$ and $p_i = 0$ for $i = 2, 3, \ldots$ we have

$$K(1+z) = \sum_{i=0}^{\infty} K_i(1, \underbrace{0, \ldots, 0}_{(i-2) \text{ terms}}) z^i.$$

Alternatively, write

$$K(1+z) = \sum_{i=0}^{\infty} b_i z^i$$

where $b_0 = 1, b_i = K_i(1, 0, \ldots, 0) \in B$. The polynomial $Q(z) = K(1+z)$ defined above is called the *characteristic power series* of the m-sequence $\{K_j\}$. Later on, we will prove that the characteristic power series does determine the m-sequence and vice-versa and therefore explains its name. To show that, we may first consider the formal factorization of a polynomial as follows,

$$1 + p_1 z + p_2 z^2 + \cdots + p_m z^m = (1 + \beta_1 z)(1 + \beta_2 z) \cdots (1 + \beta_m z).$$

We can apply an m-sequence on it by the property of being multiplicative. Since p_i can be viewed as elementary symmetric polynomials in $\beta_1, \ldots, \beta_m$. The ring $\mathcal{B}$ is then the ring of all symmetric polynomials in $\beta_1, \ldots, \beta_m$ with coefficients in B.

Proposition 1.1. *The set of m-sequences S is isomorphic to Q, which is the set of power series with constant term* 1 *and coefficients in B.*

PROOF. Define the map $f : S \longrightarrow Q$ by

$$f(\{K_j\}) = Q(1+z) = \sum_{i=0}^{\infty} b_i z^i.$$

Injectivity of f: It suffices to show that for any power series $Q(z)$ given by an m-sequence $\{K_j\}$ cannot be given by m-sequences other than $\{K_j\}$. Starting from $\{K_j\}$, we have $Q(z) = K(1+z)$. Suppose we have another m-sequence $\{K'_j\}$, which has the same characteristic power series $Q(z)$ as $\{K_j\}$, then it suffices to show that $K'(\sum_{i=0}^{m} p_i z^i) = K'(\sum_{i=0}^{m} p_i z^i)$ for any $m = 0, 1, 2, \ldots$. Consider the formal factorization of a particular polynomial to the power of $2m$,

(71) $$1 + p_1 z + \cdots + p_m z^m = \Pi_{i=1}^{m}(1 + \beta_i z).$$

Operate K' on (71),

$$\begin{aligned} K'(1 + p_1 z + \cdots + p_m z^m) &= \Pi_{i=1}^{m} K'(1 + \beta_i z) \\ \sum_{i=0}^{m} K'_i(p_1, \ldots, p_i) z^i &= \Pi_{i=1}^{m} Q(\beta_i z) = \sum_{i=0}^{m} K_i(p_1, \ldots, p_i) z^i. \end{aligned}$$

The last equality is by definition of characteristic power series. Hence we have

$$\sum_{i=0}^{m} K'_i(p_1, \ldots, p_i) z^i = \sum_{i=0}^{m} K_i(p_1, \ldots, p_i) z^i.$$

By the fact that the polynomial ring $\mathcal{B}$ is graded, we have $K'_i(p_1, \ldots, p_i) = K_i(P_1, \ldots, p_i)$ for $i = 0, 1, \ldots, m$. Since m can be chosen arbitrarily, $\{K'_j\} = \{K_j\}$.

Surjectivity of f: It suffices to show that for any power series $\sum_{i=0}^{\infty} b_i z^i, b_0 = 1, b_i \in B$, we can find an m-sequence $\{K_j\}$ such that $K(1+z) = \sum_{i=0}^{\infty} b_i z^i$. For any m, we have the formal factorization

$$1 + p_1 z + \cdots + p_m z^m = \Pi_{i=1}^{m}(1 + \beta_i z).$$

Define a polynomial $K_i(p_1, \ldots, p_i) \in \mathcal{B}_i$, for $i = 1, \ldots, m$ by

$$K_i(p_1, \ldots, p_i) z^i = \varkappa_{2i} \left[\sum_{i=0}^{\infty} b_i (\beta_i z)^i \right],$$

where $\varkappa_{2i}$ picks out all the monomials of degree $2i$. m can be chosen large enough to define K_j for all $j = 0, 1, \ldots$. To check that the $\{K_j\}$ defined above is an m-sequence, for any identity

$$1 + p_1 z + p_2 z^2 + \cdots = (1 + p'_1 z + p'_2 z^2 + \cdots)(1 + p''_1 z + p''_2 z^2 + \cdots),$$

we need to show that it is invariant after operated by K on both sides. It suffices to prove the truncated cases. That is, any identity

(72)
$$1 + p_1 z + p_2 z^2 + \cdots + p_m z^m = (1 + p'_1 z + p'_2 z^2 + \cdots + p'_n z^n)(1 + P''_1 z + p''_2 z^2 + \cdots + p''_l z^l)$$

will imply

$$\begin{aligned} & K(1+p_1z+p_2z^2+\cdots+p_mz^m) \\ = & K(1+p'_1z+p'_2z^2+\cdots+p'_nz^n)K(1+p''_1z+p''_2z^2+\cdots+p''_lz^l). \end{aligned}$$

Consider the formal factorization of the polynomials in (72),

$$\begin{aligned} 1+p_1z+p_2z^2+\cdots+p_mz^m &= (1+\beta_1z)(1+\beta_2z)\cdots(1+\beta_mz); \\ 1+p'_1z+p'_2z^2+\cdots+p'_nz^n &= (1+\beta'_1z)(1+\beta'_2z)\cdots(1+\beta'_nz); \\ 1+p''_1z+p''_2z^2+\cdots+p''_lz^l &= (1+\beta''_1z)(1+\beta''_2z)\cdots(1+\beta''_lz). \end{aligned}$$

Therefore,

$$\begin{aligned} & (1+\beta_1z)(1+\beta_2z)\cdots(1+\beta_mz) \\ = & (1+\beta'_1z)(1+\beta'_2z)\cdots(1+\beta'_nz)(1+\beta''_1z)(1+\beta''_2z)\cdots(1+\beta''_lz). \end{aligned}$$

By comparing coefficients of z^k, for $k=0,1,\ldots,m$, we have

$$\begin{aligned} & \sum_{1\le i_1<<\cdots<i_k\le m}\beta_{i_1}\beta_{i_2}\ldots\beta_{i_k} \\ = & \sum_{t=0}^{k}\sum_{1\le i_1<\cdots<i_t}\beta'_{i_1}\beta'_{i_2}\ldots\beta'_{i_t}\sum_{1\le j_1<\cdots<j_{k-t}}\beta''_{j_1}\beta''_{j_2}\ldots\beta''_{j_{k-t}}. \end{aligned}$$

Since

$$Q(\beta_1z)\cdot Q(\beta_2z)\cdots Q(\beta_mz)=\sum_i b_i(\beta_1z)^i\cdot\sum_i b_i(\beta_2z)^i\cdots\sum_i b_i(\beta_mz)^i;$$

$$\begin{aligned} K_1(p_1)z &= \varkappa_2[Q(\beta_1z)\cdot Q(\beta_2z)\cdots Q(\beta_mz)] \\ &= b_1\sum_{i=1}^{m}\beta_iz; \\ K_2(p_1,p_2)z^2 &= \varkappa_4[Q(\beta_1z)\cdot Q(\beta_2z)\cdots Q(\beta_mz)] \\ &= b_2\sum_{i=1}^{m}(\beta_i)^2z^2; \\ &\vdots \\ K_m(p_1,p_2,\ldots,p_m)z^m &= \varkappa_{2m}[Q(\beta_1z)\cdot Q(\beta_2z)\cdots Q(\beta_mz)]z^m \\ &= b_m\sum_{i=1}^{m}(\beta_i)^mz^m, \end{aligned}$$

we have

$$\begin{aligned} & K(1+p_1z+p_2z^2+\cdots+p_mz^m) \\ = & 1+K_1(p_1)z+K_2(p_1,p_2)z^2+\cdots+K_m(p_1,\ldots,p_m)z^m \\ = & 1+b_1\sum_{i=1}^{m}\beta_iz+b_2\sum_{i=1}^{m}(\beta_i)^2z^2+\cdots+b_m\sum_{i=1}^{m}(\beta_i)^mz^m. \end{aligned}$$

Similarly, since

$$\begin{aligned}
K_1(p'_1)z &= b_1\sum_{i=1}^{n}\beta'_i z;\\
K_2(p'_1,p'_2)z^2 &= b_2\sum_{i=1}^{n}(\beta'_i)^2 z^2;\\
&\vdots\\
K_n(p'_1,p'_2,\dots,p'_n)z^n &= b_n\sum_{i=1}^{n}(\beta'_i)^n z^n;
\end{aligned}$$

$$\begin{aligned}
K_1(p''_1)z &= b_1\sum_{i=1}^{l}\beta''_i z;\\
K_2(p''_1,p''_2)z^2 &= b_2\sum_{i=1}^{l}(\beta''_i)^2 z^2;\\
&\vdots\\
K_l(p''_1,p''_2,\dots,p''_l)z^l &= b_l\sum_{i=1}^{l}(\beta''_i)^l z^l,
\end{aligned}$$

then

$$\begin{aligned}
&K(1+p'_1z+p'_2z^2+\cdots+p'_nz^n)\cdot K(1+p''_1z+p''_2z^2+\cdots+p''_lz^l)\\
=\ &\left(1+b_1\sum_{i=1}^{b}\beta'_iz+b_2\sum_{i=1}^{n}(\beta'_i)^2z^2+\cdots+b_n\sum_{i=1}^{n}(\beta'_i)^nz^n\right)\\
&\cdot\left(1+b_1\sum_{i=1}^{l}\beta''_iz+b_2\sum_{i=1}^{l}(\beta''_i)^2z^2+\cdots+b_l\sum_{i=1}^{m}(\beta''_i)^lz^l\right).
\end{aligned}$$

We will show $K(1+p_1z+p_2z^2+\cdots+p_mz^m) = K(1+p'_1z+p'_2z^2+\cdots+p'_nz^n)\cdot K(1+p''_1z+p''_2z^2+\cdots+p''_lz^l)$ by induction with respect to the power of z. Compare the coefficients of z^1, we have

$$b_1\sum_{i=1}^{m}\beta_i = 1\cdot b_1\sum_{i=1}^{l}\beta''_i + b_1\sum_{i=1}^{n}\beta'_i\cdot 1.$$

This is true if we eliminate b_1 from both sides. Assume for $z^k, k\le m$, we have

$$\begin{aligned}
&b_k\sum_{i=1}^{m}(\beta_i)^k\\
=\ &\sum_{t=0}^{k}\sum_{1\le i_1<i_2<\cdots<i_t\le n} b_{i_1}\sum_{i=1}^{n}(\beta'_i)^{i_1}\cdot b_{i_2}\sum_{i=1}^{n}(\beta'_i)^{i_2}\cdots b_{i_t}\sum_{i=1}^{n}(\beta'_i)^{i_t}\\
&\cdot\sum_{s=1}^{k-t}\sum_{1\le j_1<j_2<\cdots<j_s\le l} b_{j_1}\sum_{i=1}^{l}(\beta''_i)^{j_1}\cdot b_{j_2}\sum_{i=1}^{l}(\beta''_i)^{j_2}\cdots b_{j_s}\sum_{i=1}^{l}(\beta''_i)^{j_s}.
\end{aligned}$$

For the $k+1$ cases,

$$
\begin{aligned}
& b_{k+1}\sum_{i=1}^{m}(\beta_i)^{k+1} \\
= & \sum_{t=0}^{k}\sum_{1\le i_1<\cdots<i_{t+1}\le n} b_{i_1}\sum_{i=1}^{n}(\beta_i')^{i_1} \\
& \cdot b_{i_2}\sum_{i=1}^{n}(\beta_i')^{i_2}\cdots b_{i_t}\sum_{i=1}^{n}(\beta_i')^{i_t}\cdot b_{i_{t+1}}\sum_{i=1}^{n}(\beta_i')^{i_{t+1}} \\
& \cdot\sum_{s=1}^{k-t}\sum_{1\le j_1<\cdots<j_s\le l} b_{j_1}\sum_{i=1}^{l}(\beta_i'')^{j_1}\cdot b_{j_2}\sum_{i=1}^{l}(\beta_i'')^{j_2}\cdots b_{j_s}\sum_{i=1}^{l}(\beta_i'')^{j_s} \\
& +\sum_{t=0}^{k}\sum_{1\le i_1<\cdots<i_t\le n} b_{i_1}\sum_{i=1}^{n}(\beta_i')^{i_1}\cdot b_{i_2}\sum_{i=1}^{n}(\beta_i')^{i_2}\cdots b_{i_t}\sum_{i=1}^{n}(\beta_i')^{i_t} \\
& \cdot\sum_{s=1}^{k-t}\sum_{1\le j_1<\cdots<j_s\le l} b_{j_1}\sum_{i=1}^{l}(\beta_i'')^{j_1} \\
& \cdot b_{j_2}\sum_{i=1}^{l}(\beta_i'')^{j_2}\cdots b_{j_s}\sum_{i=1}^{l}(\beta_i'')^{j_s}\cdot b_{j_{s+1}}\sum_{i=1}^{l}(\beta_i'')^{j_{s+1}} \\
= & \sum_{t=0}^{k+1}\sum_{1\le i_1<\cdots<i_t\le n} b_{i_1}\sum_{i=1}^{n}(\beta_i')^{i_1}\cdot b_{i_2}\sum_{i=1}^{n}(\beta_i')^{i_2}\cdots b_{i_t}\sum_{i=1}^{n}(\beta_i')^{i_t} \\
& \cdot\sum_{s=1}^{k-t}\sum_{1\le j_1<\cdots<j_s\le l} b_{j_1}\sum_{i=1}^{l}(\beta_i'')^{j_1}\cdot b_{j_2}\sum_{i=1}^{l}(\beta_i'')^{j_2}\cdots b_{j_s}\sum_{i=1}^{l}(\beta_i'')^{j_s}
\end{aligned}
$$

and hence it is also true for $k = k+1$. By induction, the equality holds for all $k = 1, 2, \ldots, m$. We can choose m arbitrarily large so that the $\{K_j\}$ is an m-sequence. By injectivity and surjectivity we have a one-one correspondence between the m-sequences and power series with 1 as constant term. □

Now change the coordinate by $z = x^2, \beta_i = \gamma_i^2$ and change p_i to c_i corresponding to the identity

$$\sum_{i=0}^{\infty} p_i(-z)^i = \left(\sum_{j=0}^{\infty} c_j(-x)^j\right)\left(\sum_{i=0}^{\infty} c_i x^i\right).$$

DEFINITION 1.2 (Todd Polynomials). Let the coefficient ring $B = \mathbb{Q}$, the field of rational numbers. Consider the m-sequence $\{T_k(c_1, \ldots, c_k)\}$ with characteristic power series

$$Q(x) = \frac{x}{1-e^{-x}} = e^{\frac{1}{2}x}\frac{\frac{1}{2}x}{\sinh\frac{1}{2}x}. \tag{73}$$

The polynomials T_k are called *Todd polynomials.*

1.2. Todd Genus and Generalized Todd Genus. Let M be a differentiable manifold, ξ a continuous $GL(q, \mathbb{C})$-bundle over M, with Chern classes $c_i \in H^{2i}(M, \mathbb{Z})$.

Definition 1.3 (Total Todd Classes). The *total Todd class* is defined to be

$$td(\xi) = \sum_{j=0}^{\infty} T_j(c_1, \ldots, c_j),$$

where $\{T_j(c_1, \ldots, c_j)\}$ is the Todd polynomials as defined in (73).

Alternatively, if $\sum_{j=0}^{q} c_j x^j = \Pi_{i=1}^{q}(1 + \gamma_i x)$, then $td(\xi) = \Pi_{i=1}^{q} \dfrac{\gamma_i}{1 - e^{-\gamma_i}}$. Indeed,

$$\begin{aligned} td(\xi) &= \sum_{j=0}^{\infty} T_j(c_1, \ldots, c_j) \\ &= T\left(\sum_{j=0}^{q} c_j\right) \\ &= T(\Pi_{i=1}^{q}(1 + \gamma_i) \\ &= \Pi_{i=1}^{q} T(1 + \gamma_i) \\ &= \Pi_{i=1}^{q} Q(\gamma_i) = \Pi_{i=1}^{q} \frac{\gamma_i}{1 - e^{-\gamma_i}}. \end{aligned}$$

Definition 1.4 (Todd Genus). Let M be an n-dimensional complex manifold and let $[M_n]$ be the fundamental homology class of $H_n(M, \mathbb{Z})$. Since the Chern classes $c_i(M)$ are elements in $\check{\mathrm{H}}^{2i}(M, \mathbb{Z})$, consider the product $c_{\gamma_1}(M) \cdots c_{\gamma_i}(M)$ where $\gamma_1 + \cdots + \gamma_i = n$. The value $c_{\gamma_1} \cdots c_{\gamma_i}(M)[M_n]$ is therefore an integer. Since the Todd polynomial is over the ring of rational number, we will get the rational number

$$T(M) = T_n[M_n]$$

where $T_n = T_n(c_1(M), \ldots, c_n(M))$ is a Todd polynomial. This number $T(M)$ is the *Todd genus*(or *T-genus*) of M.

Let $\varkappa_n[\quad]$ denote the sum of all homogeneous monomials of degree n in terms of the $\gamma_i's$. Note that the Todd genus of M can be written alternatively as

$$T(M) = \varkappa_n[td(\xi)][M_n] = \varkappa_n\left[\Pi_{i=1}^{q} \frac{\gamma_i}{1 - e^{-\gamma_i}}\right][M_n]$$

in terms of the their characteristic polynomials.

Definition 1.5 (Generalized Todd Genus). Now consider the m-sequence $\{T_y(j; c_1, \ldots, c_j)\}$ which is associated to the characteristic power series $Q(y; x) = \dfrac{x(y+1)}{1 - e^{-x(y+1)}} - xy$.

$$T_y(M) = T_n(y; c_1(M), \ldots, c_n(M))[M_n]$$

is a polynomial of degree n in y with rational coefficients. The polynomial $T_y(M)$ is called the *generalized Todd genus*(or the T_y-*genus*.)

Note that the generalized Todd genus of M can also be written alternatively as

$$\begin{aligned} T_y(M) &= \varkappa_n[\sum_j T_y(y; c_1(M), \ldots, c_j(M))][M_n] \\ &= \varkappa_n[\Pi_{j=1}^n Q(y;\gamma_j)][M_n] = \varkappa_n\left[\Pi_{j=1}^n \frac{\gamma_j(y+1)}{1-e^{-\gamma_i(y+1)}} - \gamma_j y\right][M_n] \end{aligned}$$

in terms of their characteristic polynomials.

Write $T_y(M)$ as a polynomial in y:

$$T_y(M) = \sum_{p=0}^n T^p(M) y^p.$$

The coefficients $T^p(M)$ in front of y^p of any p is that

$$T^p(M) = \varkappa_n\left[\sum_{j_k \text{distinct}} e^{-\gamma_{j_1}-\cdots-\gamma_{j_p}} \Pi_{i=1}^n \frac{\gamma_i}{1-e^{-\gamma_i}}\right].$$

Indeed,

$$\begin{aligned} &\sum_{p=0}^n \varkappa_n[e^{-\gamma_{j_1}-\cdots-\gamma_{j_p}} \Pi_{i=1}^n \frac{\gamma_i}{1-e^{-\gamma_i}}]y^p \\ = &\ \varkappa_n[\sum_{p=0}^n(\sum_{j_1\neq\cdots\neq j_p} e^{-\gamma_{j_1}-\cdots-\gamma_{j_p}})y^p \Pi_{i=1}^n \frac{\gamma_i}{1-e^{-\gamma_i}}] \\ = &\ \varkappa_n[(1+ye^{-\gamma_1})(1+ye^{-\gamma_2})\cdots(1+ye^{-\gamma_n})\Pi_{i=1}^n \frac{\gamma_i}{1-e^{-\gamma_i}}] \\ = &\ \varkappa_n[\Pi_{i=1}^n(1+ye^{-\gamma_i})\frac{\gamma_i}{1-e^{-\gamma_i}}] \\ = &\ \varkappa_n[\Pi_{i=1}^n(1+ye^{-(y+1)\gamma_i})\frac{\gamma_i}{1-e^{-(1+y)\gamma_i}}] \\ = &\ \varkappa_n[\Pi_{i=1}^n Q(y;\gamma_i)] \\ = &\ T_n(y; c_1, \ldots, c_n) = \sum_{p=0}^n T_n^p(c_1, \ldots, c_n) y^p. \end{aligned}$$

That is,

$$T_n^p(c_1, \ldots, c_n) = \varkappa_n\left[\sum_{j_1\neq\cdots\neq j_p} e^{-\gamma_{j_1}-\cdots-\gamma_{j_p}} \Pi_{i=1}^n \frac{\gamma_i}{1-e^{-\gamma_i}}\right].$$

1.3. Virtual Generalized Todd Genus. This subsection will firstly represent the Todd genus of a codimension 1 submanifold of the complex n dimensional manifold M by Todd-classes of M and secondly use the representation to define virtual Todd genus of codimension k submanifold. Let V be an oriented $(n-1)$-dimensional submanifold of M, then by definition the T_y-genus of V is given by

$$T_y(V) = T_{n-1}(y; c_1(V), \ldots, c_n(V))[V_{n-1}].$$

The embedding of V into M induces a bundle map as the following diagram shown:

$$\begin{array}{ccc} j^*(TM) & \xleftarrow{\mathrm{j}^*} & TM \\ \big\downarrow & & \big\downarrow \\ V & \xrightarrow{\mathrm{j}} & M \end{array}$$

Further we have the decomposition

$$j^*TM = TV \oplus \nu$$

where ν is the normal bundle over V with respect to the bundle $j^*(TM)$. That is, for any point $x \in V$ the fiber ν_x in ν over x is defined by the quotient vector space $j^*(TM)_x/TV_x$. This embedding also induces a pull back function

$$j^* : \check{\mathrm{H}}^*(TM, \mathbb{Z}) \longrightarrow \check{\mathrm{H}}^*(j^*(TM), \mathbb{Z}),$$

thus the total Chern class of M, $c(M)$, has a image $j^*c(M)$ which is the total Chern classes of the vector bundle $j^*(TM) \longrightarrow V$. By the fact that $j^*(TM) = TV \oplus \nu$, we get the identity

$$j^*c(M)(= c(j^*(TM))) = c(V_{n-1})c(\nu).$$

If further the cohomology class $v \in \check{\mathrm{H}}^1(M, \mathbb{Z})$ is the Poincaré dual of the homology class $[V_{n-1}]$ in $H_{n-1}(M, \mathbb{Z})$, we will have $c(\nu) = 1 + j^*(v)$ whose proof can be found in [Hir] p.69. Therefore,

$$\begin{aligned} & 1 + c_1(V^{n-1}) + c_2(V^{n-1}) + \cdots + c_{n-1}(V^{n-1}) \\ = \; & c(V^{n-1}) \\ = \; & j^*c(M)(c(\nu))^{-1} \\ = \; & j^*(1 + c_1(M) + c_2(M) + \cdots + c_n(M))(1 + j^*(v))^{-1} \\ = \; & j^*(1 + c_1(M) + c_2(M) + \cdots + c_n(M))(j^*(1 + v))^{-1}. \end{aligned}$$

By the property of m-sequence,

$$\begin{aligned} & T_y(1 + c_1(V^{n-1}) + c_2(V^{n-1}) + \cdots + c_{n-1}(V^{n-1})) \\ = \; & T_y(j^*(1 + c_1(M) + c_2(M) + \cdots + c_n(M))(j^*(1 + v))^{-1}) \\ = \; & T_y(j^*(1 + c_1(M) + c_2(M) + \cdots + c_n(M))(T_y(j^*(1 + v)))^{-1} \\ = \; & j^*T_y(1 + c_1(M) + c_2(M) + \cdots + c_n(M))\frac{1}{Q(y; v)} \\ = \; & j^*T_y(1 + c_1(M) + c_2(M) + \cdots + c_n(M))\frac{R(y; v)}{v}, \end{aligned}$$

where $R(y;x) = \dfrac{e^{x(y+1)}-1}{e^{x(y+1)}+y}$.

Evaluate at $[V_{n-1}]$, we get the expression of T_y-genus of V^{n-1}.

$$T_y(V^{n-1}) \;=\; j^*T_y(1+c_1(M)+c_2(M)+\cdots+c_n(M))\frac{R(y;v)}{v}[V_{n-1}].$$

Use the identity

$$j^*(x)[V_{n-1}] = (vx)[M_n], \quad x \in \check{\mathrm{H}}^{2n-2}(M,\mathbb{Z})$$

where $x = T_y(1+c_1(M)+c_2(M)+\cdots+c_n(M))\dfrac{R(y;v)}{v}$, and get

$$\begin{aligned}(74) \quad T_y(V^{n-1}) &= vT_y(1+c_1(M)+c_2(M)+\cdots+c_n(M))\frac{R(y;v)}{v}[M_n] \\ &= T_y(1+c_1(M)+c_2(M)+\cdots+c_n(M))R(y;v)[M_n],\end{aligned}$$

in which the proof can be found [Hir] p.87. We will also denote $\varkappa_n[u] = u[M_n]$ for $u \in \check{\mathrm{H}}^{2n}(M,\mathbb{Z})$, so the meaning of $\varkappa_n$ will depend on the context. Then we get,

$$\begin{aligned}T_y(V^{n-1}) &= \varkappa_n[R(y;v)T_y(1+c_1(M)+c_2(M)+\cdots+c_n(M))] \\ &= \varkappa_n[R(y;v)\sum_{j=0}^{\infty}T_j(y;c_1(M),\ldots,c_j(M))] \\ &= \varkappa_n[R(y;v)\Pi_{j=1}^{n}Q(y;\gamma_j)]\end{aligned}$$

Note that the term $R(y;v)$ is the change from $T_y(M)$ to $T_y(V^{n-1})$. As a formal generalization of the last formula, we may define the *virtual* T_y*-genus* as follows.

DEFINITION 1.6. For $v_1,\ldots,v_r \in \check{\mathrm{H}}^2(M,\mathbb{Z})$, define

$$\begin{aligned}(75) \quad T_y(v_1,\ldots,v_r)_M &= \varkappa_n[R(y;v_1)\ldots R(y;v_r)\sum_{j=0}^{\infty}T_j(y;c_1(M),\ldots,c_j(M))] \\ &= \varkappa_n[R(y;v_1)\ldots R(y;v_r)\Pi_{j=1}^{n}Q(y;\gamma_i)]\end{aligned}$$

to be the *virtual* T_y*-genus of the* r*-tuple* $(v_1,\ldots,v_r)$. The r-tuple is called the *virtual almost complex* $(n-r)$*-dimensional submanifold of* M*.*

Note that $R(y;v_1)\ldots R(y;v_r)$ appearing in Definition 1.6 is the change from $T_y(M_n)$ to $T_y(v_1,\ldots,v_r)_{M_n}$. With this definition, we know that $T_y(v_1,\ldots,v_r)_M$ is a polynomial in y of degree $(n-r)$ with coefficients in the rational numbers. Write

$$T_y(v_1,\ldots,v_r)_M = \sum_{p=0}^{n-r}T^p(v_1,\ldots,v_r)_M y^p.$$

Then the rational number $T(v_1,\ldots,v_r)_M$ defined by $T_0(v_1,\ldots,v_r)_M$ is called the *virtual Todd genus of the virtual submanifold* $(v_1,\ldots,v_r)$.

1.4. T-Characteristics and T_y-Characteristics. Analogue to all kinds of "genus" of the tangent bundle, we have the "characteristics" of an arbitrary $GL(q,\mathbb{C})$-bundle ξ over a complex n-dimensional manifold M. Fix the notations as follows: Let ξ be a complex analytic $GL(q,\mathbb{C})$-bundle over M, write

$$c(M) = \sum_{i=0}^{n} c_i; \quad c(\xi) = \sum_{i=0}^{q} d_i$$

where $c_i, d_i \in \check{\mathrm{H}}^{2i}(M,\mathbb{Z})$ are the Chern classes of TM and ξ respectively.

Definition 1.7 (Total Chern Character). The *total Chern character* of the $GL(q,\mathbb{C})$-bundle ξ is

$$ch(\xi) = \sum_{i=1}^{q} e^{\gamma_i}.$$

Proposition 1.2. *(i)* $ch(\xi \oplus \xi') = ch(\xi) + ch(\xi')$; *(ii)* $ch(\xi \otimes \xi') = ch(\xi)ch(\xi')$.

Proof. Omitted. □

Definition 1.8 (T-Characteristic). The *T-characteristic* of ξ is defined by

$$T(M,\xi) = \varkappa_n[ch(\xi)td(TM)],$$

where $ch(\xi)$ is the Chern character of ξ and $td(TM)$ is the Todd class of M.

Proposition 1.3. $T(M,\xi \oplus \xi') = T(M,\xi) + T(M,\xi')$.

Proof. By using Proposition 1.2,

$$\begin{aligned} T(M,\xi \oplus \xi') &= \varkappa_n[ch(\xi \oplus \xi')td(TM)] \\ &= \varkappa_n[(ch(\xi) + \chi(\xi'))td(TM)] \\ &= \varkappa_n[\chi(\xi)td(TM)] + \varkappa_n[\chi(\xi')td(TM)] = T(M,\xi) + T(M,\xi') \end{aligned}$$

□

To define the T_y-characteristic of ξ, we will write it as a polynomial in y as $T_y(M,\xi) = \sum_p T^p(M,\xi)y^p$ and define $T^p(M,\xi)$ first. Considering the dual tangent bundle T^*M, its Chern classes are the elementary symmetric functions in $-\gamma_i, 1 \leq i \leq n$. Considering the bundle $\wedge^p(T^*M)$, its Chern classes are the elementary symmetric functions in $-(\gamma_{j_1} + \gamma_{j_2} + \cdots + \gamma_{j_p})$. We now define the number $T^p(M,\xi)$ by $T(M, \wedge^p(T^*M) \otimes \xi)$, which is the T-characteristic of the vector bundle $\wedge^p(T^*M) \otimes \xi$. That is,

$$\begin{aligned} T^p(M_n,\xi) &= \varkappa_n[ch(\wedge^p T^*M \otimes \xi)td(T^*M)] \\ &= \varkappa_n[ch(\xi)ch(\wedge^p(T^*M)td(T^*M)]. \end{aligned}$$

Hence we have:

DEFINITION 1.9 (T_y-Characteristic). The T_y*-characteristic* of ξ over M is defined by

$$T_y(M,\xi) = \sum_{p=0}^{n} T^p(M,\xi)y^p,$$

where $T^p(M,\xi) = \varkappa_n[ch(\xi)ch(\wedge^p(T^*M)td(TM)]$.

That is,

$$\begin{aligned}
T_y(M,\xi) &= \sum_{p=0}^{n} T^p(M,\xi)y^p \\
&= \sum_{p=0}^{n} \varkappa_n[ch(\xi)ch(\wedge^p(T^*M)\Pi_{i=1}^n Q(\gamma_i)]y^p \\
&= \sum_{p=0}^{\infty} \varkappa_n \left[\sum_{i=1}^{q} e^{\delta_i} \sum_{i_1\neq\cdots\neq i_p} e^{-\gamma_{i_1}-\cdots-\gamma_{i_p}} \Pi_{i=1}^n \frac{\gamma_i}{1-e^{-\gamma_i}}\right] y^p \\
&= \sum_{p=0}^{\infty} \varkappa_n \left[\sum_{i=1}^{q} e^{\delta_i} \Pi_{j=1}^n (1+ye^{-\gamma_j})\Pi_{i=1}^n \frac{\gamma_i}{1-e^{-\gamma_i}}\right] y^p \\
&= \varkappa_n \left[\sum_{j=1}^{q} e^{\delta_j} \Pi_{i=1}^n (1+ye^{-\gamma_i})\frac{\gamma_i}{1-e^{-\gamma_i}}\right] \\
&= \varkappa_n \left[\sum_{j=1}^{q} e^{(1+y)\delta_j} \Pi_{i=1}^n (1+ye^{-(1+y)\gamma_i})\frac{\gamma_i}{1-e^{-(1+y)\gamma_i}}\right] \\
&= \varkappa_n[\sum_{j=1}^{q} e^{(1+y)\delta_j}\Pi_{i=1}^n Q(y;\gamma_i)] \\
&= \varkappa_n[(\sum_{j=1}^{q} e^{(1+y)\delta_j})(\sum_{j=0}^{n} T_j(y;c_1,\ldots,c_j))].
\end{aligned}$$

I.e.,

$$T_y(M_n,\xi) = \varkappa_n[ch_{(y)}(\xi)\Pi_{i=1}^n Q(y;\gamma_i)];\ \text{or}$$

$$T_y(M_n,\xi) = \varkappa_n[ch_{(y)}(\xi)(\sum_{j=0}^{n} T_j(y;c_1,\ldots,c_j))],$$

where we have denoted the polynomial $\sum_{j=1}^{q} e^{(1+y)\delta_j}$ by $ch_{(y)}(\xi)$.

1.5. Virtual T_y-Characteristic. As a formal generalization of the formula (75) of the virtual T_y-genus of M_n, we have the virtual T_y-characteristic.

DEFINITION 1.10 (Virtual T_y-characteristics). Let M be a complex manifold of dimension n. Let ξ be a $GL(q,\mathbb{C})$-bundle over M and $v_1,\ldots,v_r \in \check{\mathrm{H}}^2(M,\mathbb{Z})$, then the *virtual T_y-characteristic of ξ with respect to the virtual submanifold* $(v_1, \ldots, v_r)$ is defined by

$$T_y(v_1, \ldots, v_r|, \xi)_M = \varkappa_n \left[ch_{(y)}(\xi)\Pi_{i=1}^r R(y; v_i) \sum_{j=0}^{\infty} T_j(y; c_1(M), \ldots, c_j(M)) \right].$$

Note that $\Pi_{i=1}^r R(y; v_i)$ is the change from $T_y(M, \xi)$ to $T_y(v_1, \ldots, v_r|, \xi)_M$.

CHAPTER 6

Riemann-Roch Theorem

This chapter is devoted to the proof of the Riemann-Roch theorem for vector bundles over algebraic manifold. In the first section, we introduce the split method and the split manifold, which is associated to any compact complex manifold. In the second section, we give the Riemann-Roch theorem for the associated split manifold of a given compact complex manifold, since both the Euler-Poincaré characteristics and the Todd-characteristics can be easily computed for a split manifold. In the third section, we give the equivalence of Todd-characteristic of a compact complex manifold and its associated split manifold, and also the equivalence of the Euler-Poincaré characteristic of a compact manifold and its associated split manifold. Therefore, we can give the Riemann-Roch theorem for algebraic manifolds. In the last section, we will give the Riemann-Roch theorem for both line bundles and vector bundles over an algebraic manifold.

1. Split Methods and Split Manifolds

1.1. Split Methods. Let M be a complex manifold, the group $G = GL(q, \mathbb{C})$. Let ξ be a G-bundle over M. Let $P \longrightarrow M$ be the principal bundle with structure group and fiber G associated to the G-bundle ξ. Let $G' = \Delta(q, \mathbb{C})$ be the subgroup of the group G. Therefore the induced bundle

$$P/G' \xrightarrow{\rho} M$$

is a fiber bundle with fiber $G/G' = GL(q, \mathbb{C})/\Delta(q, \mathbb{C}) = F(q)$, the manifold of flags and with the structure group G. As shown in the commutative diagram:

$$\begin{array}{ccc} P & \longrightarrow & P/G' \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

Induce the bundle $\rho^* P \longrightarrow P/G'$ from $P \longrightarrow M$ by the mapping $\rho : P/G' \longrightarrow M$. The corresponding G-bundle $\rho^*\xi$ can be reduced to a $\Delta(q, \mathbb{C})$-bundle, since $P \longrightarrow P/G'$ is a principal bundle with structure group and fiber $\Delta(q, \mathbb{C})$. Let $\xi_1, \xi_2, \ldots, \xi_q$ be the associated diagonal $\mathbb{C}^*$-bundles of $\rho^*\xi$ over P/G' as shown in the diagram:

$$
\begin{array}{ccccc}
 & & \rho^* P & & \\
 & \overset{\rho^*}{\nearrow} & & \overset{\rho^*\xi}{\nwarrow} & \\
P & & \longrightarrow & & P/G' \\
 & \searrow & & \overset{\rho}{\swarrow} & \\
 & & M & &
\end{array}
$$

Let $T(q) \longrightarrow F(q)$ be the tangent bundle of the manifold of flags with fiber $GL(m, \mathbb{C})$ where $m = \dfrac{q(q-1)}{2}$. Since $GL(q, \mathbb{C})$ acts on $F(q)$, then $GL(q, \mathbb{C})$ also acts on $T(q)$. From the principal bundle $P \longrightarrow M$ with structure group and fiber G and the associated G-bundle ξ, we want to induce a fiber bundle $\mathfrak{G}(q) \longrightarrow M$ with fiber $T(q)$, structure group G and the same associated G-bundle ξ. Simply define

$$\mathfrak{G}(q) = P \times T(q)/ \sim$$

where $\sim$ is an equivalence relation given by $(p \times at) \sim (pa \times t)$ whenever $a \in G$ for all $p \in P, t \in T(q)$. The commutative diagram is given as follows,

$$
\begin{array}{ccccc}
\mathfrak{G}(q) & \longrightarrow & P & \longrightarrow & P/G' \\
 & \searrow & \downarrow & \swarrow & \\
 & & M & &
\end{array}
$$

Consider the diagram

$$
\begin{array}{ccc}
\mathfrak{G}(q) & \overset{\theta}{\longrightarrow} & P/G' \\
 \searrow & & \swarrow \\
 & M &
\end{array}
$$

The bundle map $\theta : \mathfrak{G}(q) \longrightarrow P/G'$ between bundles over M is given by $T(q) \mapsto F(q)$ over any $x \in M$. Considering $\theta : \mathfrak{G}(q) \longrightarrow P/G'$ as a fiber bundle, it is a principal bundle with structure group and fiber $GL(m, \mathbb{C})$ where $m = \dfrac{q(q-1)}{2}$. Let ξ^{Δ} denote its associated $GL(m, \mathbb{C})$-bundle, then ξ^{Δ} is called the *bundle along fibers $F(q)$ of P/G'*. Consider the induced bundle $\rho^*\mathfrak{G}(q) \longrightarrow P/G'$ from the fiber bundle $\mathfrak{G}(q) \longrightarrow M$ by the mapping $\rho : P/G' \longrightarrow M$ as shown in the following diagram. This is a fiber bundle associated to the $GL(m, \mathbb{C})$-bundle $\rho^*\xi$, which is equal to $h\xi^{\Delta}$ where h is the inclusion from $\Delta(m, \mathbb{C})$-bundles over P/G' to $GL(m, \mathbb{C})$-bundles over P/G'. Hence $\rho^*\xi$ can be reduced to a $\Delta(m, \mathbb{C})$-bundle. Hence there are m diagonal $\mathbb{C}^*$-bundles over P/G' associating to it, which is the m diagonal $\mathbb{C}^*$-bundles associated to ξ^{Δ}. The following theorem describes the m $\mathbb{C}^*$-bundles by the q $\mathbb{C}^*$-bundles of ρ^*P.

$$\begin{array}{ccccc} & & \rho^*\mathfrak{G}(q) & & \\ & \overset{\rho^*}{\nearrow} & & \overset{\rho^*\xi}{\searrow} & \\ \mathfrak{G}(q) & & \xrightarrow{\xi^\Delta} & & P/G' \\ & \overset{\xi}{\searrow} & & \overset{\rho}{\swarrow} & \\ & & M & & \end{array}$$

Theorem 1.1 (Main Theorem). *With the above notations: The m diagonal $\mathbb{C}^*$-bundle of the $GL(m;\mathbb{C})$-bundle ξ^Δ is $\xi_i \otimes \xi_j^{-1}$ in the following order: $\xi_i \otimes \xi_j^{-1}$ is before $\xi_{i'} \otimes \xi_{j'}^{-1}$ if and only if either $j > j'$ or $j = j', i < i'$.*

Proof. The theorem is trivially true when $q = 1$, we assume the theorem is true for $q - 1$.

Step 1. To construct the following diagram:

$$\begin{array}{ccccccccc} \mathfrak{G}(q) & & \xrightarrow{\xi^\Delta} & & P/G' & & \xleftarrow{\overline{\xi^\Delta}} & & \mathfrak{G}(q-1) \\ & \overset{T(q)}{\searrow} & & \overset{F(q)}{\swarrow} & & \overset{F(q-1)}{\searrow} & & \overset{T(q-1)}{\swarrow} & \\ & & M & & \xleftarrow{\mathbb{P}_{\mathbb{C}}^{q-1}} & & \overline{M} & & \end{array}$$

We can induce from $P/G' \longrightarrow M$ a fiber bundle $\overline{M} = (P/G')/F(q) \xrightarrow{\bar{\rho}} M$ in the canonical way so that $P/G' \longrightarrow \overline{M}$ is a fiber bundle with fiber $F(q-1)$ and structure group $GL(q-1,\mathbb{C})$, since the kernel of the homomorphism $GL(1, q-1;\mathbb{C}) \longrightarrow GL(q-1)$ acts trivially on $F(q-1) = GL(q-1,\mathbb{C})/\Delta(q-1,\mathbb{C}) = GL(1, q-1;\mathbb{C})/\Delta(q,\mathbb{C})$. From the following diagram,

$$\begin{array}{ccccc} P/G' & & \xrightarrow{\bar{\rho}} & & \overline{M} \\ & \overset{\rho}{\searrow} & & \overset{\psi}{\swarrow} & \\ & & M & & \end{array}$$

the fiber bundle $\overline{M} \xrightarrow{\psi} M$ has typical fiber

$$\begin{aligned} F(q)/F(q-1) &= (GL(q,\mathbb{C})/\Delta(q,\mathbb{C}))/(GL(1,q-1;\mathbb{C})/\Delta(q,\mathbb{C})) \\ &= GL(q,\mathbb{C})/GL(1,q-1;\mathbb{C}) = \mathbb{P}_{\mathbb{C}}^{q-1}, \end{aligned}$$

structure group G and is associated to the G-bundle ξ. Induce the fiber bundle $\psi^*(P/G') \longrightarrow \overline{M}$ by the map $\overline{M} \xrightarrow{\psi} M$ from the fiber bundle $P/G' \longrightarrow M$ as shown in the diagram

$$\begin{array}{ccccc} & & \psi^*P/G' & & \\ & \overset{\psi^*}{\nearrow} & & \searrow & \\ P/G' & & \xrightarrow{\bar{\rho}} & & \overline{M} \\ & \overset{\rho}{\searrow} & & \overset{\psi}{\swarrow} & \\ & & M & & \end{array}$$

The associated $GL(1, q-1; \mathbb{C})$-bundle $\psi^*\xi$ has the decomposition $\psi^*\xi = (\eta, \bar{\xi})$ where η is a $\mathbb{C}^*$-bundle and $\bar{\xi}$ is a $GL(q-1, \mathbb{C})$ bundle. Induce the fiber bundle $\bar{\rho}^*\psi^* P/G' \longrightarrow P/G'$ by the map $P/G' \xrightarrow{\bar{\rho}} \overline{M}$ from the fiber bundle $\psi^* P/G' \longrightarrow \overline{M}$ as shown in the following diagram

$$\begin{array}{ccccc} & & \bar{\rho}^*\psi^* P/G' & & \\ & \swarrow & & \overset{\bar{\rho}^*}{\nwarrow} & \\ P/G' & & & & \psi^* P/G' \\ & \overset{\bar{\rho}}{\searrow} & & \swarrow & \\ & & \overline{M} & & \end{array}$$

The associated G-bundle $\bar{\rho}^*\psi^*\xi = \bar{\rho}^*(\eta, \bar{\xi}) = (\bar{\rho}^*\eta, \bar{\rho}^*\bar{\xi})$ where $\bar{\rho}^*\eta$ is a $\mathbb{C}^*$-bundle and $\bar{\rho}^*\bar{\xi}$ is a $GL(q-1; \mathbb{C})$-bundle. In the other direction, consider the following diagram:

$$\begin{array}{ccccc} & & \rho^* P & & \\ & \overset{\rho^*}{\nearrow} & & \searrow & \\ P & & \overset{\xi^\Delta}{\longrightarrow} & & P/G' \\ & \searrow & & \overset{\rho}{\swarrow} & \\ & & M & & \end{array}$$

Recall that the induced associated G-bundle $\rho^*\xi$ has the diagonal $\mathbb{C}^*$-bundles $\xi_1, \xi_2, \ldots, \xi_q$. We have the commutativity $\rho = \psi \circ \bar{\rho}$:

$$\begin{array}{ccccc} P/G' & & \overset{\bar{\rho}}{\longrightarrow} & & \overline{M} \\ & \overset{\rho}{\searrow} & & \overset{\psi}{\swarrow} & \\ & & M & & \end{array}$$

Hence, $\rho^*\xi = \bar{\rho}^*\psi^*\xi$, and therefore

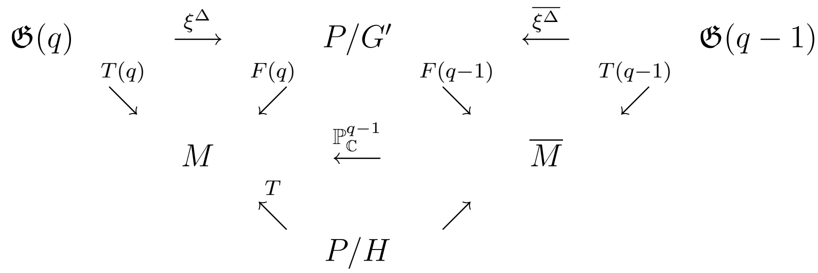
$$\begin{aligned} \rho^*\xi &= \bar{\rho}^*\psi^*\xi \\ (\xi_1, \xi_2, \ldots, \xi_q) &= (\bar{\rho}^*\eta, \bar{\rho}^*\bar{\xi}). \end{aligned}$$

That is, $\bar{\rho}^*\eta = \xi_1$ and $\bar{\rho}^*\bar{\xi} = (\xi_2, \ldots, \xi_q)$. Let $T(q-1) \longrightarrow F(q-1)$ be the tangent bundle. Since the group $GL(q-1, \mathbb{C})$ acts on $F(q-1)$, it also acts on $T(q-1)$. By Using the $GL(q-1)$-bundle $\bar{\xi}$, we can construct a fiber bundle $\mathfrak{G}(q-1) \xrightarrow{\overline{\xi^\Delta}} \overline{M}$ with fiber $T(q-1)$ and associated to $\bar{\xi}$ as shown in the diagram:

$$\begin{array}{ccccc} P/G' & & \overset{\overline{\xi^\Delta}}{\longleftarrow} & & \mathfrak{G}(q-1) \\ & \overset{F(q-1)}{\searrow} & & \overset{T(q-1)}{\swarrow} & \\ & & \overline{M} & & \end{array}$$

By the assumption of induction, since $\bar{\xi}$ has the $(q-1)$ diagonal $\mathbb{C}^*$-bundles $\xi_2, \ldots, \xi_q$, then $\overline{\xi^\Delta}$ has the $\bar{m} = \dfrac{(q-1)(q-2)}{2}$ diagonal $\mathbb{C}^*$-bundles $\xi_i \otimes \xi_j^{-1}$ for $2 \leq i, j \leq q$ in the following order: $\xi_i \otimes \xi_j^{-1}$ is before $\xi_{i'} \otimes \xi_{j'}^{-1}$ if and only if either $j > j'$ or $j = j', i < i'$.

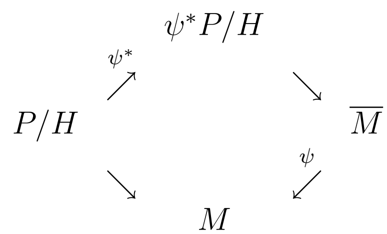
Step 2. To construct the following diagram:



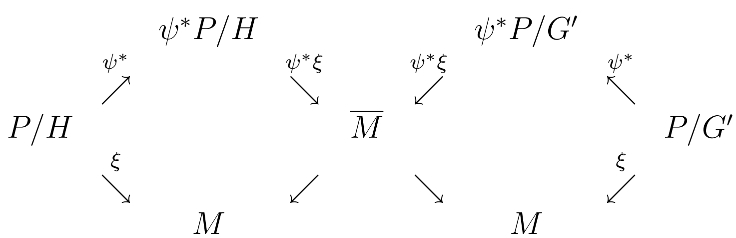
Let $T \longrightarrow \mathbb{P}_{\mathbb{C}}^{q-1}$ be the principal tangent bundle with structure group and fiber $GL(q-1, \mathbb{C})$. The group G acts on $\mathbb{P}_{\mathbb{C}}^{q-1}$ and hence G acts on T. Construct a fiber bundle $Q \longrightarrow M$ with fiber T structure group G and is associated to the G-bundle ξ. Since $Q \longrightarrow \overline{M}$ has $GL(q-1, \mathbb{C})$ as typical fiber, then $Q = \overline{M}/GL(q-1, \mathbb{C}) = (P/GL(1, q-1; \mathbb{C}))/GL(q-1, \mathbb{C}) = P/H$ where $H = GL(1, q-1; \mathbb{C})/GL(q-1; \mathbb{C})$ which is set of matrices in the form

$$\begin{pmatrix} a & a_{12} \dots & a_{1q} \\ 0 & aI & \end{pmatrix}$$

with I the $(q-1) \times (q-1)$ identity matrix. Thus $P/H \longrightarrow \overline{M}$ is a principal bundle with structure group and fiber $GL(q-1, \mathbb{C})$. Now induce the fiber bundle $\psi^* P/H \longrightarrow \overline{M}$ by the map $\psi : \overline{M} \longrightarrow M$ from the fiber bundle $P/H \longrightarrow M$ as shown in the diagram:



To give the associated $GL(q-1, \mathbb{C})$-bundle $\psi^*\xi$, consider the following commutative diagram:



The associated $GL(1, q-1; \mathbb{C})$-bundle ξ of $P/H \longrightarrow M$ can be reduced to a $GL(q-1, \mathbb{C})$-bundle by the map $h : GL(1, q-1; \mathbb{C}) \longrightarrow GL(q-1; \mathbb{C})$ where $h : A \mapsto a^{-1}A''$, since H is the kernel of h. On the other hand, the associated bundle $\psi^*\xi$ of $\psi^* P/G' \longrightarrow M$ is given by $(\eta, \overline{\xi})$. Using the fact that the two $\psi^*\xi$'s are equivalent, the associated bundle $\psi^*\xi$ of $\psi^* P/H \longrightarrow \overline{M}$ is equal to $(\eta, \overline{\xi})$. When the ξ is reduced to a $GL(q-1, \mathbb{C})$-bundle by h, the $\psi^*\xi = (\eta, \overline{\xi})$ can also reduce to a $GL(q-1; \mathbb{C})$-bundle $\eta^{-1} \otimes \overline{\xi}$. If we denote the associated $GL(q-1)$-bundle of $P/H \longrightarrow \overline{M}$ to be a, then a is equal to $\eta^{-1} \otimes \overline{\xi}$.

Step 3. Induce $\overline{\rho}^* P/H \longrightarrow P/G'$ by the map $\overline{\rho} : P/G' \longrightarrow \overline{M}$ from the fiber bundle $P/H \longrightarrow \overline{M}$ as shown in the diagram

$$\begin{array}{ccccc} & & P/G' & & \\ & \nearrow & & \overset{\overline{\rho}}{\searrow} & \\ \overline{\rho}^* P/H & & & & \overline{M} \\ & \overset{\overline{\rho}^*}{\nwarrow} & & \nearrow & \\ & & P/H & & \end{array}$$

Then the associated $GL(q-1, \mathbb{C})$-bundle $\overline{\rho}^* a = \overline{\rho}^*(\eta^{-1} \otimes \overline{\xi})$ is reduced to a $\Delta(q-1, \mathbb{C})$-bundle naturally and has $(q-1)$ diagonal $\mathbb{C}^*$-bundle $\xi^{-1} \otimes \xi_2, \xi^{-1} \otimes \xi_3, \ldots, \xi^{-1} \otimes \xi_q$. Together with the diagram constructed in step 2, commutativity of $\rho = \psi \circ \overline{\rho}$ gives

$$\xi^\Delta = (\overline{\xi^\Delta}, \overline{\rho}^* a),$$

hence the ξ^Δ has the $\overline{m} + (q-1) = m$ diagonal $\mathbb{C}^*$-bundles as required. □

1.2. Split Manifolds. As the first application of Theorem 1.1, we will calculate some total Chern classes. Further we will give the notion of split manifold. Let M be a complex manifold of complex dimension n, $TM \longrightarrow M$ be the tangent bundle of M with the associated $GL(n, \mathbb{C})$-bundle ξ. Let $P \longrightarrow M$ be the principal bundle with structure group and fiber $GL(n, \mathbb{C})$ and associated to ξ. Let $G' = \Delta(n, \mathbb{C})$. Denote $P/G' \xrightarrow{\rho} M$ by $M^\Delta \xrightarrow{\rho} M$, which is associated to the same ξ. Let $TM^\Delta \longrightarrow M^\Delta$ be the induced bundle with structure group $GL(m, \mathbb{C})$ where $m = \dfrac{n(n-1)}{2}$, such that over each $x \in M$, $TM^\Delta_x \longrightarrow M^\Delta_x$ is the tangent bundle of $F(n)$. Let ξ^Δ denote the associated $GL(m, \mathbb{C})$-bundle of TM^Δ. Observe that the fiber bundle $TM^\Delta \longrightarrow M^\Delta$ has ξ^Δ as subbundle and the $\rho^* TM$ as quotient bundle. That is,

$$TM^\Delta = \xi^\Delta \oplus \rho^* TM.$$

Let $(\xi_1, \ldots, \xi_n)$ be the diagonal $\mathbb{C}^*$-bundles of $\rho^*\xi$ associated to $P \longrightarrow M^\Delta$. Write $c(\xi_i) = 1 + \gamma_i$, then by using Theorem 1.1,

$$\begin{aligned} c(M^\Delta) &= c(\xi^\Delta) c(\rho^* TM) \\ &= c((\xi_{i_1} \otimes \xi_{j_1}^{-1}) \oplus (\xi_{i_1} \otimes \xi_{j_1}^{-1}) \oplus \cdots \oplus (\xi_{i_n} \otimes \xi_{j_n}^{-1})) c(\rho^*\xi) \\ &= c(\xi_{i_1} \otimes \xi_{j_1}^{-1}) \cdot c(\xi_{i_1} \otimes \xi_{j_1}^{-1}) \cdots c(\xi_{i_n} \otimes \xi_{j_n}^{-1}) \cdot c(\rho^*\xi) \\ &= \Pi_{1 \le j < i \le n}(1 + \gamma_i - \gamma_j) c(\rho^*\xi) \\ &= \Pi_{1 \le j < i \le n}(1 + \gamma_i - \gamma_j) \Pi_{i=1}^n (1 + \gamma_i) \end{aligned}$$

That is,

$$c(M^\Delta) = \Pi_{i=1}^n (1 + \gamma_i) \Pi_{1 \le j < i \le n}(1 + \gamma_i - \gamma_j). \tag{76}$$

Definition 1.1 (Complex Analytic Split Manifold). A complex manifold M of complex dimension n is called a *complex analytic split manifold* if the associated complex analytic $GL(n, \mathbb{C})$-bundle ξ admits the group $\Delta(n, \mathbb{C})$ as complex analytic structure groups, that is ξ is an element in the image of the inclusion

$$H^1(X, \Delta(n,\mathbb{C})_\omega) \longrightarrow H^1(X, GL(n,\mathbb{C})_\omega).$$

This ξ is associated to n diagonal $\mathbb{C}^*$-bundles $\xi_1, \ldots, \xi_n \in H^1(X, \mathbb{C}^*_\omega)$. If all bundles are regarded as differentiable bundles the ξ is the Whitney sum $\xi = \xi_1 \oplus \cdots \oplus \xi_n$ and further if $c(\xi_i) = 1 + a_i$ then $c(M) = c(\xi) = \Pi_{i=1}^n (1 + a_i)$.

REMARK 1.1. Note that each M^Δ constructed above is in particular a split manifold. That is, the above constructions associated each manifold M a split manifold M^Δ. In the later, we will show that certain theorems hold on M whenever they are hold on M^Δ hence it will suffice to work on the associated split manifolds.

2. Riemann-Roch Theorem for Split Manifolds

2.0.1. *Basic Theorem I.*

DEFINITION 2.1 (Algebraic Manifold). A compact complex manifold M is called an *algebraic manifold* if it admits a complex holomorphic embedding as a submanifold of a complex projective space of some dimension.

THEOREM 2.1 (Basic Theorem I). *Let M be an algebraic manifold, $E \longrightarrow M$ be a complex holomorphic vector bundle and let $F_1, \ldots, F_r$ be complex holomorphic line bundles over M. Denote such set by $(F_1, \ldots, F_r|, E)_M$. Let G be a function which associates to each $(F_1, \ldots, F_r)_M$ a rational number which is independent of the order in which F_i appear. Suppose that for some fixed rational number y_0,*

(I) $G(M) = \chi_{y_0}(M)$

(II) $G(F_1, \ldots, G_r, A \otimes B) = G(F_1, \ldots, F_r, A)_M + G(F_1, \ldots, F_r, B)_M + (y_0 - 1)G(F_1, \ldots, F_r, A, B)_M - y_0 G(F_1, \ldots, F_r, A, B, A \otimes B)_M$.

(III) *If S is a non-singular divisor of M and $F_1 = \{S\}$ then $G(F_1, \ldots, F_r)_M = G((F_2)_S, \ldots, (F_r)_S)_S$, where for $r = 1$, $G(F_1)_M = G(S)$. If $F_1 = \{0\} = 1$ then $G(F_1, \ldots, F_r)_M = 0$.*

Then, For all $(F_1, \ldots, F_r)_M$ with $r \geq 1$,

$$\chi_{y_0}(F_1, \ldots, F_r)_M = G(F_1, \ldots, F_r)_M.$$

Before proving Theorem 2.1, we need a theorem of Kodaira for *algebraic manifolds*, refer to $[Hir]$ p. 141.

THEOREM 2.2. *Let M be an algebraic manifold and F a complex holomorphic line bundle over M. Then F can always be written in the form $F = \{S\} \otimes \{T\}^{-1}$ where S and T are non-singular divisors of M.*

PROOF. (of Theorem 2.1) We will prove the theorem by induction on the dimension of the M.

Step 0. When M is of 0 dimensional. Suppose M is consists of k points, then

$$\chi_{y_0}(M,E) = \chi(M,E) = dim H^0(M,E) = qk;$$
$$\chi_{y_0}(F_1,\ldots,F_r|,E) = 0 \text{ for } r \geq 1$$

are satisfied. On the other hand, by (III), $G(F_1,\ldots,F_r,E)_M = 0$. Hence $\chi_{y_0}(F_1,\ldots,F_r) = G(F_1,\ldots,F_r)_M$ is shown for M is zero dimensional. Suppose the theorem is proved for M of dimension less than n, and suppose in the following that M is of complex dimension n.

Step 1. χ_y has property (II): Write $u = w\Pi_{i=1}^r \hat{R}(f_i)$, then it is equivalent to prove

$$\begin{aligned}\hat{h}_M(u\hat{R}(ab)) &= \hat{h}_M(u\hat{R}(a)) + \hat{h}_M(u\hat{R}(b)) + (y-1)\hat{h}_M(u\hat{R}(a)\hat{R}(b)) \\ &\quad -y\hat{h}_M(u\hat{R}(a)\hat{R}(b)\hat{R}(ab))\end{aligned}$$

where $\hat{R}(x) = \dfrac{1-x^{-1}}{1+yx^{-1}}$. Since $\hat{h}$ is a d-homomorphism, then it suffices to show that

$$\hat{R}(ab) = \hat{R}(a) + \hat{R}(b) + (y-1)\hat{R}(a)\hat{R}(b) - y\hat{R}(a)\hat{R}(b)\hat{R}(ab). \tag{77}$$

The equation (77) is satisfied as an purely algebraic property when $\hat{R}(x) = \frac{e^{cx}-1}{e^{cx}+y}$ for any indeterminate c and y. (Refer to [Hir] p. 95.) Hence χ_y has (II). In particular when $y = y_0$, χ_{y_0} has (II).

Step 2. χ_y has property (III): By definition,

$$\chi_y((F_2)_S,\ldots,(F_r)_S|,E_S)_S = \hat{h}_S(w\Pi_{i=2}^r \hat{R}(f_i))$$

where $\hat{R}(x) = \dfrac{1-x^{-1}}{1+yx^{-1}}$. Claim: $\hat{h}_S(w^\mu f_1^{\lambda_1}\cdots f_r^{\lambda_r}) = \hat{h}_M(w^\mu f_1^{\lambda_1}\cdots f_r^{\lambda_r}\hat{R}(f_1))$.
Proof of claim:

$$\begin{aligned}\hat{h}_S(w^\mu f_1^{\lambda_1}\cdots f_r^{\lambda_r}) &= \chi_y(S, E^\mu\otimes F_1^{\lambda_1}\otimes\cdots\otimes F_r^{\lambda_r}) \\ &= \chi_y(S,E^\mu)\cdot\chi_y(S,F_1^{\lambda_1}\otimes\cdots\otimes F_r^{\lambda_r}) \\ &= \hat{h}_M\left(w^\mu\frac{1-s^{-1}}{1+ys^{-1}}\right)\cdot\hat{h}_M(\Pi_{i=1}^r\hat{R}(\lambda_i)) \\ &= \hat{h}_M(w^\mu\Pi_{i=1}^r\hat{R}(f_1)\cdot\hat{R}(\lambda_i)) = \hat{h}_M(w^\mu f_1^{\lambda_1}\cdots f_r^{\lambda_r}\hat{R}(f_1)).\end{aligned}$$

So this is the claim. Hence,

$$\begin{aligned}\chi_y((F_2)_S,\ldots,(F_r)_S|,E_S)_S &= \hat{h}_S(w\Pi_{i=2}^r\hat{R}(f_i)) \\ &= \hat{h}_M(w\Pi_{i=1}^r\hat{R}(f_i)) = \chi_y(F_1,\ldots,F_r|,w)_M.\end{aligned}$$

Hence, χ_y satisfies (III). In particular when $y = y_0$, χ_{y_0} also satisfies (III).

By Step 1 and Step 2, χ_{y_0} satisfies (II) and (III). By letting $G' = \chi_{y_0} - G$, the statement in the theorem can be translated into: For any function G' which associates to each $(F_1, \dots, F_r|, E)_M$ a power series, if G' satisfies (II) and (III) and further

(I′) $G'(M, E) = 0$,

then G' is identically zero.

Step 3. Prove $G' = 0$: By Theorem 2.2, for the holomorphic line bundle F_1, there exist non-singular divisors S and T of M such that $\{S\} = F_1 \otimes \{T\}$. then by property (II),

$$\begin{aligned} & G'(\{S\}, F_2, \dots, F_r|, E) \\ = \; & G'(F_1, \dots, F_r|, E) + G'(\{T\}, F_2, \dots, F_r|, E) \\ & + (y_0 - 1) G(\{T\}, F_1, \dots, F_r|, E) - y_0 G'(\{S\}, \{T\}, F_1, \dots, F_r|, E) \end{aligned}$$

Hence,

$$\begin{aligned} & G'(F_1, \dots, F_r|, E) \\ = \; & \underbrace{G'(\{S\}, F_2, \dots, F_r|, E)}_{A} - \underbrace{G'(\{T\}, F_2, \dots, F_r|, E)}_{B} \\ & -(y_0 - 1) \underbrace{G(\{T\}, F_1, \dots, F_r|, E)}_{C} + y_0 \underbrace{G'(\{S\}, \{T\}, F_1, \dots, F_r|, E)}_{D}. \end{aligned}$$

It suffices to show that A, B, C, D all equal to zero. When $r > 1$, by condition (III) and the induction assumption,

$$\begin{aligned} A &= G'(F_{2S}, \dots, F_{rS}|, E)_S = 0; \\ B &= G'(F_{2T}, \dots, F_{rT}|, E)_T = 0; \\ C &= G'(F_{1T}, F_{2T}, \dots, F_{rT}|, E)_T = 0; \\ D &= G'(\{T\}_S, F_{1S}, F_{2S}, \dots, F_{rS}|, E)_S = 0. \end{aligned}$$

When $r = 1$, by condition (I′),

$$\begin{aligned} A &= G'(\{S\}, E) = 0; \\ B &= G'(\{T\}|, E) = 0. \end{aligned}$$

By (III) and induction assumption,

$$\begin{aligned} C &= G'(\{T\}, F_1|, E) = G'(F_{1T}|, E)_T| = 0; \\ D &= G'(\{S\}, \{T\}, F_1|, E) = G'(\{T\}_S, F_{1S}|, E)_S = 0. \end{aligned}$$

Hence $G'(F_1, \dots, F_r|, E) = 0$. □

2.0.2. *Basic Theorem II.* Consider the set $(F_1, \dots, F_r|, E)_M$ let $f_i \in H^2(M, \mathbb{Z})$ be the cohomology class of F_i, $i = 1, \dots, r$ and ξ be the cohomology class of E, then we write the virtual T_y-characteristics $T_y(f_1, \dots, f_r|, \xi)_M$ by $T_y(F_1, \dots, F_r|, E)_M$ and also write $T_y(M, E)$ for $T_y(M, \xi)$.

THEOREM 2.3 (Basic Theorem II). *Let M be an algebraic manifold of complex dimension n and let $F_1, \dots, F_r$ be complex holomorphic line bundles over M with cohomology classes $f_1, \dots, f_r \in H^2(M, \mathbb{Z})$. Then*

$$\chi_1(F_1, \dots, F_2)_M = T_1(F_1, \dots, F_r)_M$$

PROOF. We will show the theorem by checking the conditions (I), (II) and (III) in Theorem 2.1 for $y_0 = 1$.

(I) $T_1(M) = \chi_1(M)$: Use the fact $T_1(M) = \tau(M)$ from [Hir], which is proven by using the cobordism theory, and the Hodge index theorem, $\chi_1(M) = \tau(M)$. $T_1(M) = \chi_1(M)$ is shown.

(II): By the algebraic facts that for c, y being indeterminates and $R(x) = \frac{e^{cx}-1}{e^{cx}+y}$, then

$$R(u+v) = R(u) + R(v) + (y-1)R(u)R(v) - yR(u)R(v)R(u+v). \tag{78}$$

Recall the definition of virtual T_y-genus of M,

$$T_y(v_1, \ldots, v_r)_M = \varkappa_n \left[R(y; v_1) \cdots R(y; v_r) \sum_{j=0}^{\infty} T_j(y; c_1(M), \ldots, c_j(M)) \right],$$

where $R(y; x) = \dfrac{e^{x(y+1)} - 1}{e^{x(y+1)} + y}$ will satisfy the functorial property (78) by substituting $c = y + 1$. Let A and B be any vector bundle over M associated to $a, b \in H^2(M, \mathbb{Z})$ respectively. Then,

$$\begin{aligned}
&T_y(F_1, \ldots, F_r, A \otimes B) \\
= \;& \varkappa_n[R(y; f_1) \cdots R(y; f_r) R(y; a+b) \sum_{j=0}^{\infty} T_j(y; c_1(M), \ldots, c_j(M))] \\
= \;& \varkappa_n[R(y; f_1) \cdots R(y; f_r)(R(y; a) + R(y; b) + (y-1)R(y; a)R(y; b) \\
& -yR(y; a)R(y; b)R(y; a+b)) \sum_{j=0}^{\infty} T_j(y; c_1(M), \ldots, c_j(M))] \\
= \;& T_y(F_1, \ldots, F_r, A) + T_y(F_1, \ldots, F_r, B) \\
& +(y-1)T_y(F_1, \ldots, F_r, A, B) - yT_y(F_1, \ldots, F_r, A, B, A \otimes B).
\end{aligned}$$

When $y = 1$, we have,

$$\begin{aligned}
T_1(F_1, \ldots, F_r, A \otimes B) \;=\;& T_1(F_1, \ldots, F_r, A) + T_1(F_1, \ldots, F_r, B) \\
& -T_1(F_1, \ldots, F_r, A, B, A \otimes B).
\end{aligned}$$

This gives (II).

(III): Let S be a non-singular divisor of M and $F_1 = \{S\}$. Let $j : S \hookrightarrow M$ denote the inclusion. By identity (74),

$$\sum_{i=0}^{\infty} T_i(y; c_1(S), \ldots, c_i(S)) = j^* \left(\frac{R(y; v)}{v} \sum_{i=0}^{\infty} T_j(y; c_1(M), \ldots, c_j(M) \right),$$

we have $T_y(j^*F_2, \ldots, j^*F_r)_S = T_y(\{S\}, F_2, \ldots, F_r)_M$. In particular, when $y = 1$,

$$T_1(F_1, F_2, \ldots, F_r)_M = T_1(F_{2S}, \ldots, F_{rS})_S.$$

Hence (III) is shown. By Theorem 2.1, we finally have $\chi_1(F_1, \ldots, F_r)_M = T_1(F_1, \ldots, F_r)_M$. □

2.0.3. *Riemann-Roch Theorem for Algebraic Split Manifolds.*

LEMMA 2.1. *Let M be a complex analytic split manifold of dimension n and let $A_1, \dots, A_n$ denote the complex analytic diagonal line bundles over M. Then*

$$(1+y)^n T(M) = \sum_{l=0}^{n} y^l \sum_{1 \le i_1 < \cdots < i_l \le n} T_y(a_{i_1}, \dots, a_{i_l})_M.$$

PROOF.

$$\begin{aligned}
&\sum_{l=0}^{n} y^l \sum_{1 \le i_1 < \cdots < l_l \le n} T_y(a_{i_1}, \dots, a_{i_l})_M \\
= \;& \sum_{l=0}^{n} y^l \sum_{1 \le i_1 < \cdots < i_l \le n} \varkappa_n \left[R(y; a_{i_1}) \dots R(y; a_{i_l}) \sum_{j=0}^{\infty} T_j(y; c_1(M), \dots, c_j(M)) \right] \\
= \;& \sum_{1 \le i_1 < \cdots < i_l \le n} \varkappa_n \left[(\sum_{l=0}^{n} (\sum_{1 \le i_1 < \cdots < i_l \le n} R(y; a_{i_1}) \dots R(y; a_{i_l})) y^l \sum_{i=1}^{n} Q(y; a_i) \right] \\
= \;& \varkappa_n [\Pi_{i=1}^n (1 + yR(y; a_i)) \cdots \Pi_{i=1}^n Q(y; a_i)] \\
= \;& \varkappa_n [\Pi_{i=1}^n (1 + yR(y; a_i)) Q(y; a_i)] \\
= \;& \varkappa_n [\Pi_{i=1}^n (Q(y; a_i) + y a_i)] \quad (\text{ by } Q(y; x) = \frac{x}{R(y; x)}) \\
= \;& \varkappa_n \left[\Pi_{i=1}^n \left(\frac{a_i(y+1)}{1 - e^{-a_i(y+1)}} - y a_i + y a_i \right) \right] \quad (\text{ by } Q(y; x) = \frac{x(y+1)}{1 - e^{-x(y+1)}} - yx) \\
= \;& \varkappa_n \left[\Pi_{i=1}^n \frac{a_i(y+1)}{1 - e^{-a_i(y+1)}} \right] \\
= \;& (1+y)^n \varkappa_n \left[\Pi_{i=1}^n \frac{a_i}{1 - e^{-a_i}} \right] = (1+y)^n T(M)
\end{aligned}$$

□

LEMMA 2.2. *Let M be a complex analytic split manifold of dimension n and let $A_1, \dots, A_n$ denote the complex analytic diagonal line bundles over M. Then*

$$(1+y)^n \chi(M, E) = \sum_{l=0}^{n} y^l \sum_{i_1 < \cdots < i_l} \chi_y(A_{i_1}, \dots, A_{i_l}|, E)_M.$$

And hence in particular when E is the trivial line bundle,

$$(1+y)^n \chi(M) = \sum_{l=0}^{n} y^l \sum_{i_1 < \cdots < i_l} \chi_y(A_{i_1}, \dots, A_{i_l})_M.$$

PROOF. Step 1. Show the identity $\chi_y(M, E) = h(w \Pi_{i=1}^n (1 + y a_i^{-1}))$: Consider the vector bundle $\wedge^p T \longrightarrow M$ it admits $\Delta(\binom{n}{p}, \mathbb{C})$ as structure group and by Theorem 1.1 the corresponding $\binom{n}{p}$ diagonal complex line bundles are given by $A_{i_1}^{-1} \otimes A_{i_2}^{-1} \otimes \cdots \otimes A_{i_p}^{-1}$ where $1 \leq i_1 < \cdots < i_p \leq n$. Thus $\wedge^p T = \oplus_{i_1 < \cdots < i_p}(A_{i_1}^{-1} \otimes A_{i_2}^{-1} \otimes \cdots \otimes A_{i_p}^{-1})$. Hence,

$$\begin{aligned} \chi^p(M, E) &= \chi(M, E \otimes \wedge^p(T)) \\ &= \sum_{i_1 < \cdots < i_p} \chi(M, E \otimes A_{i_1}^{-1} \otimes \cdots \otimes A_{i_p}^{-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} \chi_y(M, E) &= \sum_{p=0}^{\infty} \chi^p(M, E) y^p \\ &= \sum_{p=0}^{\infty} \sum_{i_1 < \cdots < i_p} \chi(M, E \otimes A_{i_1}^{-1} \otimes \cdots \otimes A_{i_p}^{-1}) y^p \\ &= \sum_{p=0}^{\infty} \sum_{i_1 < \cdots < i_p} h(w a_{i_1}^{-1} \ldots a_{i_p}^{-1}) y^p \\ &= h\left(w \sum_{p=0}^{\infty} \sum_{i_1 < \cdots < i_p} (a_{i_1}^{-1} \ldots a_{i_p}^{-1}) y^p \right) = h(w \Pi_{i=1}^{n} (1 + y a_i^{-1})), \end{aligned}$$

that is $\chi_y(M, E) = h(w \Pi_{i=1}^{n} (1 + y a_i^{-1}))$.

Step 2. Compute

$$\begin{aligned} & \sum_{l=0}^{n} y^l \sum_{i_1 < \cdots < i_l} \chi_y(A_{i_1}, \ldots, A_{i_l}|, E)_M \\ = & \sum_{l=0}^{n} y^l \sum_{i_1 < \cdots < i_l} \hat{h}\left(w \frac{1 - a_{i_1}^{-1}}{1 + y a_{i_1}^{-1}} \cdots \frac{1 - a_{i_l}^{-1}}{1 + y a_{i_l}^{-1}} \right) \\ = & \sum_{l=0}^{n} y^l \sum_{i_1 < \cdots < i_l} \hat{h}(w \hat{R}(a_{i_1}) \hat{R}(a_{i_2}) \cdots \hat{R}(a_{i_l})) \\ = & \hat{h}\left(\sum_{l=0}^{n} y^l \sum_{i_1 < \cdots < i_l} w \hat{R}(a_{i_1}) \hat{R}(a_{i_2}) \cdots \hat{R}(a_{i_l}) \right) \\ = & \hat{h}\left(w \sum_{l=0}^{n} y^l \sum_{i_1 < \cdots < i_l} \hat{R}(a_{i_1}) \hat{R}(a_{i_2}) \cdots \hat{R}(a_{i_l}) \right) \\ = & \hat{h}(w \Pi_{i=1}^{n} (1 + y \hat{R}(a_i))) \\ = & \hat{h}\left(w \Pi_{i=1}^{n} \frac{1 + y}{1 + y a_i^{-1}} \right) = (1 + y)^n \hat{h}\left(w \Pi_{i=1}^{n} \frac{1}{1 + y a_i^{-1}} \right), \end{aligned}$$

where $\hat{R}(x) = \dfrac{1-x^{-1}}{1+yx^{-1}}$. By the identity obtained in Step 1, we have

$$\begin{aligned}
\hat{h}(w^\mu a_1^{\lambda_1} a_2^{\lambda_2} \dots a_r^{\lambda_r}) &= \hat{h}(w^\mu) \cdot \hat{h}(a_1^{\lambda_1} \dots a_r^{\lambda_r}) \\
&= \chi_y(M, E^\mu) \cdot \chi_y(M, A_1^{\lambda_1} \otimes \dots \otimes A_r^{\lambda_r}) \\
&= h(w\Pi_{i=1}^n (1+ya_i^{-1}) \cdot h(a_1^{\lambda_1} \dots a_r^{\lambda_r}) \\
&= h(w^\mu a_1^{\lambda_1} a_2^{\lambda_2} \dots a_r^{\lambda_r} \Pi_{i=1}^n (1+ya_i^{-1})).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{l=0}^n y^l \sum_{i_1<\dots<i_l} \chi_y(A_{i_1}, \dots, A_{i_l}|, E)_M \\
= \; & (1+y)^n \hat{h}\left(w\Pi_{i=1}^n \frac{1}{1+ya_i^{-1}} \right) \\
= \; & (1+y)^n h(w \cdot (\Pi_{i=1}^m (1+ya_i^{-1})^{-1})(\Pi_{i=1}^n (1+ya_i^{-1}))) \\
= \; & (1+y)^n h(w) = (1+y)^n \chi(M, E).
\end{aligned}$$

□

THEOREM 2.4 (Riemann-Roch Theorem for Algebraic Split Manifolds). *Let M be an algebraic manifold which is also a complex analytic split manifold. Then*

$$\chi(M) = T(M).$$

PROOF. By Lemma 2.1 and Lemma 2.2,

$$(1+y)^n T(M) = \sum_{l=0}^n y^l \sum_{1 \le i_1 < \dots < l_l \le n} T_y(a_{i_1}, \dots, a_{i_l})_M;$$

$$(1+y)^n \chi(M) = \sum_{l=0}^n y^l \sum_{i_1 < \dots < i_l} \chi_y(A_{i_1}, \dots, A_{i_l})_M.$$

We will use the fact saying that for algebraic manifold M the χ_y is also a polynomial, refered to [Hir] p. 142. The proof uses the similar induction process as that in the proof of Theorem 2.1. Since T_y is by definition a polynomial, to show $T(M)$ and $\chi(M)$ are the same, it suffices to show they agree for an y_0 which is not equal to -1. By Theorem 2.3, we have the polynomials $T(M)$ and $\chi(M)$ agree when $y_0 = 1$. Hence the result. □

3. Riemann-Roch Theorem for Algebraic Manifolds

3.1. $T(M) = T(M^\Delta)$**.** The subsection will gives the proof of $T(M) = T(M^\Delta)$. That is, the Todd genuses of two differentiable manifolds agree whenever the Todd genuses of their associated split manifolds agree. Hence it suffices to show the case of split manifolds in the proof of the Riemann-Roch Theorem.

3.1.1. *Algebraic Preliminary.* Let K be a field of characteristic 0 and $c_1, \ldots, c_n$ be the indeterminates, thus we have an extension field $K(c_1, \ldots, c_n)$ of K. Let x be an indeterminate, suppose there are $\gamma_1, \ldots, \gamma_n$ such that

$$1 + c_1 x + \cdots + c_n x^n = (1 + \gamma_1 x) \cdots (1 + \gamma_n x).$$

Then we further get an extension field $K(c_1, \ldots, c_n)(\gamma_1, \ldots, \gamma_n)$. We will state without proof the following algebraic facts, (ref [Hir p. 107]).

THEOREM 3.1. *$K(c_1, \ldots, c_n)(\gamma_1, \ldots, \gamma_n)$ is an algebraic extension of the field $K(c_1, \ldots, c_n)$ of extension degree $n!$. Further, the $n!$ elements $\{\gamma_1^{a_1}\gamma_2^{a_2} \ldots \gamma_{n-1}^{a_{n-1}}\}$ with $0 \leq a_i \leq n - i$ forms a basis for $K(c_1, \ldots, c_n)(\gamma_1, \ldots, \gamma_n)$ over the field $K(c_1, \ldots, c_n)$.*

Thus, under the chosen basis $\{\gamma_1^{a_1}\gamma_2^{a_2} \ldots \gamma_{n-1}^{a_{n-1}}\}_{0 \leq a_i \leq n-i}$, any power series P in $\gamma_1, \ldots, \gamma_n$ with coefficients in K as an element in the field $K(c_1, \ldots, c_n)(\gamma_1, \ldots, \gamma_n)$ can be uniquely represented by

$$P = \sum_{0 \leq a_i \leq n-i} \rho_{a_1, a_2, \ldots, a_{n-1}} \gamma_1^{a_1}\gamma_2^{a_2} \ldots \gamma_{n-1}^{n-1},$$

where $\rho_{a_1, a_2, \ldots, a_{n-1}} \in K(c_1, \ldots, c_n)$. Define the *indicator of P* by

$$\sigma(P) = (-1)^{\frac{n(n-1)}{2}} \rho_{a_1, a_2, \ldots, a_{n-1}}|_{a_1 = n-1, a_2 = n-2, \ldots, a_{n-1} = n-1},$$

that is, the coefficient of the highest degree monomial $\gamma_1^{n-1}\gamma_2^{n-2} \ldots \gamma_{n-1}^1$ in P, multiplicated by $(-1)^{\frac{n(n-1)}{2}}$.

Denote $s(P)$ by the polynomial obtained from permuting indeterminates $\gamma_1, \ldots, \gamma_n$ of P. That is, $s(P) = \sum_{0 \leq a_i \leq n-i} \rho_{a_1, a_2, \ldots, a_{n-1}} \gamma_{s(1)}^{a_1}\gamma_{s(2)}^{a_2} \cdots \gamma_{s(n-1)}^{n-1}$. We will now give a formula of $\sigma(P)$ whose proof is in [Hir] p. 108.

$$\sigma(P) = \frac{(\sum_s sign(s) \cdot s(P))}{\Pi_{i>j}(\gamma_i - \gamma_j)}. \tag{79}$$

By using (79), it can be computed that

$$\sigma(\Pi_{i>j}(\gamma_i - \gamma_j)) = n!; \tag{80}$$

$$\sigma\left(\Pi_{i>j} \frac{\gamma_j - \gamma_i}{e^{\gamma_j - \gamma_i} - 1}\right) = 1. \tag{81}$$

3.1.2. $T(M) = T(M^\Delta)$. Let $F(q) = GL(q, \mathbb{C})/\Delta(q, \mathbb{C})$ be the flag manifold. $F(q) \xrightarrow{\rho} M$ can be regarded as a complex analytic split manifold over a point M. Suppose corresponding $\Delta(q, \mathbb{C})$ bundle has the q diagonal $\mathbb{C}^*$-bundles $\xi_1, \ldots, \xi_q$). Then as in (76), the total Chern class of $F(q)$ is given by $c(F(q)) = \rho^*(c(M))\Pi_{1 \leq j < i \leq q}(1 + \gamma_i - \gamma_j)$. Since M is a point, then $\rho^*(c(M)) = 1$ and thus,

$$c(F(q)) = \Pi_{1 \leq j < i \leq q}(1 + \gamma_i - \gamma_j).$$

where γ_i is given by $c(\xi_i) = 1 + \gamma_i, i = 1, \ldots, n$. We will use the following theorem of Borel, cited in [Hir] p. 108.

THEOREM 3.2. *The cohomology ring*

$$H^*(F(q),\mathbb{Z}) = \mathbb{Z}[\gamma_1,\ldots,\gamma_q]/I^+(c_1,\ldots,c_q),$$

where $c_i = c_i(F(q)$ for $i = 1,\ldots,q$ and $I^+(c_1,\ldots,c_q)$ is the ideal generated by the elementary symmetric functions $c_1,\ldots,c_q$ in the γ_i's.

Hence the γ_i's will satisfy $1 + c_1x + \cdots + c_qx^q = (1+\gamma_1)\cdots(1+\gamma_q)$. By Theorem 3.1, the $q!$ elements $\{\gamma_1^{a_1}\gamma_2^{a_2}\cdots\gamma_{q-1}\}_{0\le a_i\le q-i}$ forms an additive basis of $H^*(F(q),\mathbb{Z})$ over $\mathbb{Z}(c_1,\ldots,c_q)$. In particular, the group $H^m(F(q),\mathbb{Z})$ where $m = \dfrac{q(q-1)}{2}$ will have the generator $\gamma_1^{q-1}\gamma_2^{q-2}\cdots\gamma_{q-1}$ up to a constant in the ring $\mathbb{Z}(c_1,\ldots,c_q)$.

LEMMA 3.1. *The generator $(-1)^{\frac{q(q-1)}{2}}\gamma_1^{q-1}\gamma_2^{q-2}\cdots\gamma_{q-1}$ of the ring $H^m(F(q),\mathbb{Z})$ where $m = \dfrac{q(q-1)}{2}$ satisfies*

$$(-1)^{\frac{q(q-1)}{2}}\gamma_1^{q-1}\gamma_2^{q-2}\cdots\gamma_{q-1}[F(q)] = 1 \tag{82}$$

where $[F(q)]$ is the fundamental homology class of $F(q)$.

PROOF. It can be computed that $c_q[F(q)] = q!$ (ref. [Hir] p. 109). Also by using (80), we have,

$$\begin{aligned}
q! &= c_q[F(q)]\\
&= \varkappa_q(c(F(q))[F(q)]\\
&= \varkappa_q[\Pi_{i>j}(1+\gamma_i-\gamma_j)][F(q)]\\
&= \Pi_{i>j}(\gamma_i-\gamma_j)[F(q)]\\
&= \sigma(\Pi_{i>j}(\gamma_i-\gamma_j))(-1)^{\frac{q(q-1)}{2}}\gamma_1^{q-1}\gamma_2^{q-2}\cdots\gamma_{q-1}[F(q)]\\
&= q!(-1)^{\frac{q(q-1)}{2}}\gamma_1^{q-1}\gamma_2^{q-2}\cdots\gamma_{q-1}[F(q)].
\end{aligned}$$

That is, $\gamma_1^{q-1}\gamma_2^{q-2}\cdots\gamma_{q-1}[F(q)] = 1$. □

LEMMA 3.2. *The Todd genus of $F(q)$, $T(F(q)) = 1$.*

PROOF. By using (81) and (90), compute:

$$\begin{aligned}
T(F(q)) &= \Pi_{i=1}^q Q(1+\gamma_i)[F(q)]\\
&= \Pi_{i=1}^q \frac{\gamma_i}{1-e^{-\gamma_i}}[F(q)]\\
&= \varkappa_q[\Pi_{i=1}^q \frac{\gamma_i}{1-e^{-\gamma_i}}][F(q)]\\
&= \sigma(\Pi_{i=1}^q \frac{\gamma_i}{1-e^{-\gamma_i}})(-1)^{q(q-1)/2}\gamma_1^{q-1}\gamma_2^{q-2}\cdots\gamma_{q-1}[F(q)]\\
&= 1\cdot 1 = 1.
\end{aligned}$$

□

THEOREM 3.3. *Let M be a compact n-dimensional almost complex manifold and ξ be a differentiable $GL(q,\mathbb{C})$-bundle over it. Let P be the principal bundle with fiber and structure group $GL(q,\mathbb{C})$ associated to ξ. Induce from P a fiber bundle $P/\Delta(q,\mathbb{C})$ over M with fiber the flag manifolds $F(q)$, structure group $GL(q,\mathbb{C})$ and associated to the same ξ. Note that the flag manifold $F(q)$ can be viewed as an almost complex manifold of dimension $n+\frac{1}{2}q(q-1)$. Let ζ be a $GL(l,\mathbb{C})$-bundle over M, and consider the following induced bundle by the projection $\rho : P/\Delta(q,\mathbb{C}) \longrightarrow M$:*

$$\begin{array}{ccc} \rho^*\zeta & \xrightarrow{\rho^*} & \zeta \\ \downarrow & & \downarrow \\ P/\Delta(q,\mathbb{C}) & \xrightarrow{\rho} & M. \end{array}$$

Then the T-characteristic of ζ satisfies

$$T(P/\Delta(q,\mathbb{C}),\rho^*\zeta) = T(M,\zeta)T(F(q)) = T(M,\zeta). \tag{83}$$

More generally, if $v_1,\ldots,v_r \in H^2(M,\mathbb{Z})$, then the virtual generalized T-characteristic satisfies

$$T(\rho^*v_1,\ldots,\rho^*v_r|,\rho^*\zeta)_{P/\Delta(q,\mathbb{C})} = T(v_1,\ldots,v_r|,\zeta)_M. \tag{84}$$

PROOF. Since identity (83) is a particular case of identity (84), it suffices to show identity (84). By definition of the virtual generalized T-characteristic, let $m = \frac{q(q-1)}{2}$ then $dim P/\Delta(q,\mathbb{C}) = n+m$ and write $c(\rho^*\xi) = 1+\rho^*c_1(\xi)+\cdots+\rho^*c_q(\xi) = \Pi_{i=1}^{q}(1+\gamma_i)$, then by $TE = \rho^*TM \oplus \xi^{\Delta}$ and Theorem 1.1, we have $c(P/\Delta(q,\mathbb{C})) = \rho^*c(M)\Pi_{i>j}(1+\gamma_i-\gamma_j)$. That is, $c_k(P/\Delta(q,\mathbb{C})) =$

$\rho^* c_k(M)\Pi_{i>j}(1+\gamma_i-\gamma_j)]$. Hence,

$$\begin{aligned}
& T(\rho^* v_1, \ldots, \rho^* v_r|, \rho^*\zeta)_{P/\Delta(q,\mathbb{C})} \\
= \;& \varkappa_{m+n}[ch_{(y)}(\rho^*\zeta)\Pi_{i=1}^r R(y;\rho^* v_i)\sum_{j=0}^{n+m} T_j(y; c_1(\Delta(q,\mathbb{C})), \ldots, c_j(\Delta(q,\mathbb{C}))]|_{y=0} \\
= \;& \varkappa_{m+n}[ch(\rho^*\zeta)\Pi_{i=1}^r \frac{e^{\rho^* v_i}-1}{e^{\rho^* v_i}}\sum_{j=0}^{n+m} T_j(y; c_1(\Delta(q,\mathbb{C})), \ldots, c_j(\Delta(q,\mathbb{C}))]|_{y=0} \\
= \;& \varkappa_{m+n}[ch(\rho^*\zeta)\Pi_{i=1}^r (1-e^{-\rho^* v_i}) \\
& \sum_{k=0}^{n+m} T_k(\rho^* c_1(M)\Pi_{j<i}(1+\gamma_i-\gamma_j), \ldots, \rho^* c_k(M)\Pi_{j<i}(1+\gamma_i-\gamma_j)] \\
= \;& \varkappa_{m+n}[ch(\rho^*\zeta)\Pi_{i=1}^r (1-e^{-\rho^* v_i}) \\
& \sum_{k=0}^{n+m} T_k(\rho^* c_1(M), \ldots, \rho^* c_k(M))\sum_{k=0}^{n+m} T_k(\Pi_{j<i}(1+\gamma_i-\gamma_j))] \\
= \;& \varkappa_{m+n}[ch(\rho^*\zeta)\Pi_{i=1}^r (1-e^{-\rho^* v_i})\rho^* td(V)\cdot \Pi_{i>j}\sum_{k=0}^{n+m} T_k(1+\gamma_i-\gamma_j)] \\
= \;& \varkappa_{m+n}[ch(\rho^*\zeta)\Pi_{i=1}^r (1-e^{-\rho^* v_i})\rho^* td(M)\cdot \Pi_{j<i} Q(\gamma_i-\gamma_j)] \\
= \;& \varkappa_{n+m}\left[\rho^*\left(ch\zeta\cdot\Pi_{i=1}^r(1-e^{-v_i})\cdot td(M)\right)\Pi_{j<i}\frac{\gamma_j-\gamma_i}{e^{\gamma_j-\gamma_i}-1}\right]
\end{aligned}$$

Let $P=\Pi_{j<i}\frac{\gamma_j-\gamma_i}{e^{\gamma_j-\gamma_i}-1}$ then since $1+\rho^* c_1(\xi)+\cdots+\rho^* c_q(\xi)=\Pi_{i=1}^q(1+\gamma_i)$, Then by Theorem 3.1, P can be uniquely write down as

$$P=\sum_{0\le a_i\le q-i}\rho_{a_1\cdots a_{q-1}}\gamma_1^{a_1}\gamma_2^{a_2}\cdots\gamma_{q-1}^{a_{q-1}},$$

where each $\rho_{a_1\cdots a_{q-1}}$ is a polynomial in $\rho^*(c_i), i=1,\ldots,q$ with coefficients in $\mathbb{Z}$. Write $\rho^*\left(ch\zeta\cdot\Pi_{i=1}^r(1-e^{-v_i})\cdot td(M)\right)=\rho^* A$, then by using (81),

$$\begin{aligned}
& T(\rho^* v_1, \ldots, \rho^* v_r|, \rho^*\zeta)_{P/\Delta(q,\mathbb{C})} \\
= \;& \varkappa_{n+m}[\rho^*(A)\cdot P] \\
= \;& \varkappa_{n+m}[\rho^*(A)(-1)^{q(q-1)/2}\sigma(P)\gamma_1^{q-1}\gamma_2^{q-2}\cdots\gamma_{q-1}] \\
= \;& \varkappa_{n+m}[\rho^*(A)(-1)^{q(q-1)/2}\gamma_1^{q-1}\gamma_2^{q-2}\cdots\gamma_{q-1}].
\end{aligned}$$

Since any term $\rho^* x$ with $x\in H^*(M,\mathbb{Z})$ is zero if it has degree $>n$, hence to get terms of degree $(n+m)$, we need terms of degree n from $\rho^*(A)$ and terms of degree m from P. Hence,

$$\varkappa_{n+m}[\rho^*(A)\cdot P]_{P/\Delta(q,\mathbb{C})}=\varkappa_n[A]([M_n])\cdot\varkappa_m[\sigma(P)(-1)^m\gamma_1^{q-1}\cdots\gamma_{q-1}]([F(q)].$$

By Lemma 3.1 and (81),

$$\varkappa_m[\sigma(P)(-1)^m\gamma_1^{q-1}\cdots\gamma_{q-1}]([F(q)]=1.$$

Hence

$$\varkappa_{n+m}[\rho^*(A)\cdot P]_{P/\Delta(q,\mathbb{C})}=\varkappa_n[A]([M_n])=T(v_1,\ldots,v_q|,\zeta)_M.$$

□

If we in particular choose the $P/\Delta(q,\mathbb{C})$ to be the associated split manifold M^Δ of the manifold M and ζ to be the associated $GL(n,\mathbb{C})$-bundle to the tangent bundle of M. Then by (83) in Theorem 3.3, $T(M) = T(M^\Delta)\cdot T(F(n))$. By Lemma 3.2, $T(F(n)) = 1$, we have the agreement of Todd genus of M and its associated split manifold: $T(M) = T(M^\Delta)$.

3.2. $\chi(M) = \chi(M^\Delta)$**.** We will only state without proof the following theorem whose proof is in [Hir] p.148.

THEOREM 3.4. *Let ξ be a complex holomorphic $GL(q,\mathbb{C})$-bundle over the algebraic manifold M. Let M' be the fiber bundle associated to ξ with the flag manifold $F(q)$ as fiber. Then M' is an algebraic manifold and $\chi(M) = \chi(M')$.*

Hence if ξ is the associated $GL(q,\mathbb{C})$-bundle of TM, then M' is the split manifold M^Δ associated to the manifold M and the theorem immediately gives the agreement of the Euler-Poincaré characteristic of M and its associated split manifold: $\chi(M) = \chi(M^\Delta)$.

THEOREM 3.5 (Riemann-Roch Theorem for Algebraic Manifolds). *Let M be an algebraic manifold, then $\chi(M) = T(M)$.*

PROOF. Let M^Δ be the associated split manifold of M, then by Theorem 3.3, $T(M) = T(M^\Delta)$. By Theorem 3.4, $\chi(M) = \chi(M^\Delta)$. By Theorem 2.4, $T(M^\Delta) = \chi(M^\Delta)$. Hence $T(M) = \chi(M)$. □

4. Riemann-Roch Theorem of Vector Bundles

LEMMA 4.1. *Let M be a compact complex manifold. Let E be a vector bundle over M and $F_1,\dots,F_r$ be holomorphic line bundles over M . Let S be a non-singular divisor of M, $\{S\} = F_1$, then*

$$\chi_y(F_1,\dots,F_r|,E)_M = \chi_y((F_2)_S,\dots,(F_r)_S|,E_S)_S. \tag{85}$$

Therefore, if $L \longrightarrow M$ is a holomorphic line bundle. Then $\chi(L)_M = \chi(M) - \chi(M,L^{-1})$.

PROOF. Claim: $\hat{h}_S(w^\mu f_1^{\lambda_1}\cdots f_r^{\lambda_r}) = \hat{h}_M(w^\mu f_1^{\lambda_1}\cdots f_r^{\lambda_r}\hat{R}(f_1))$, where $\hat{R}(x) = \dfrac{1-x^{-1}}{1+yx^{-1}}$. Pf of claim:

$$\begin{aligned}
\hat{h}_S(w^\mu f_1^{\lambda_1}\cdots f_r^{\lambda_r}) &= \chi_y(S, E^\mu\otimes F_1^{\lambda_1}\otimes\cdots\otimes F_r^{\lambda_r})\\
&= \chi_y(S,E^\mu)\cdot\chi_y(S,F_1^{\lambda_1}\otimes\cdots\otimes F_r^{\lambda_r})\\
&= \hat{h}_M\left(w^\mu\frac{1-s^{-1}}{1+ys^{-1}}\right)\cdot\hat{h}_M(\Pi_{i=1}^r\hat{R}(\lambda_i))\\
&= \hat{h}_M(w^\mu\Pi_{i=1}^r\hat{R}(f_1)\cdot\hat{R}(\lambda_i)) = \hat{h}_M(w^\mu f_1^{\lambda_1}\cdots f_r^{\lambda_r}\hat{R}(f_1)).
\end{aligned}$$

Compute,

$$\begin{aligned}\chi_y((F_2)_S,\ldots,(F_r)_S|,E_S)_S &= \hat{h}_S(w\Pi_{i=2}^r\hat{R}(f_i)) \\ &= \hat{h}_M(w\Pi_{i=1}^r\hat{R}(f_i)) = \chi_y(F_1,\ldots,F_r|,w)_M.\end{aligned}$$

□

REMARK 4.2. Observe that if some F_i is trivial then by (85),

$$\chi_y(F_1,\ldots,F_r|,E)M = 0.$$

LEMMA 4.3. *Let M be a compact complex manifold of complex dimension n and $L \longrightarrow M$ a holomorphic line bundle, then $T(L)_M = T(M) - T(M, L^{-1})$.*

PROOF. The $\mathbb{C}^*$-bundle over M are in one-one correspondence to the elements in the cohomology group $H^2(M,\mathbb{Z})$. Suppose the $\mathbb{C}^*$-bundle L is associated to a $\mathbb{C}^*$-bundle η and has total Chern class $c(F) = 1 + c_1(F)$. Denote $c_1(F)$ by v_1, then by definition, the total Todd class of F is

$$T(L) = T(M,\eta) = \varkappa_n[ch(\eta)td(TM)] = \varkappa_n[e^{v_1}td(TM)].$$

On the other hand,

$$\begin{aligned}T(M) - T(L^{-1})_M &= T(M) - T(-v_1)_M \\ &= \varkappa_n[td(TM) - R(0;-v_1)td(TM)] \\ &= \varkappa_n\left[td(TM))(1-\frac{e^{-v_1}-1}{e^{-v_1}})\right] = \varkappa_n[td(TM)e^{v_1}].\end{aligned}$$

Compare the two sides and get $T(F) = T(M) - T(F^{-1})_M$. □

THEOREM 4.1 (Riemann-Roch Theorem of Line Bundles). *Let M be an algebraic manifold of complex dimension n and let L be a complex holomorphic line bundle over M with cohomology class $\eta \in H^2(M,\mathbb{Z})$. The Euler characteristic*

$$\chi(M,L) = \sum_{i=0}^{n}(-1)^i dim H^i(M,L)$$

is equal to

$$T(M,L) = \varkappa_n\left[e^{\eta}\Pi_{i=1}^n\frac{\gamma_i}{1-e^{-\gamma_i}}\right],$$

where the γ_i's is determined by the formal factorization $1 + c_1x + \cdots + c_nx^n = (1+\gamma_1x)\ldots(1+\gamma_nx)$ for $c_i \in H^{2i}(M,\mathbb{Z})$.

PROOF. By Theorem 3.5, $\chi(M) = T(M)$, that is $\chi_0(M) = T_0(M)$. By Theorem 2.1, if $F_1,\ldots,F_r$ are complex holomorphic line bundles over M, then $\chi(F_1,\ldots,F_r)_M = T(F_1,\ldots,F_r)_M$. In particular if $r = 1$ and in the case of the holomorphic line bundle $L \longrightarrow M$, we have, $\chi(L)_M = T(L)_M$. By Lemma 4.1 and Lemma 4.3, we have $\chi(L)_M = \chi(M) - \chi(M,L^{-1})$ and $T(L)_M = T(M) - T(M,L^{-1})$, respectively. Replace L by L^{-1}, we get, $\chi(L^{-1})_M = \chi(M) - \chi(M,L)$ and $T(L^{-1})_M = T(M) - T(M,L)$. Since $\chi(L^{-1})_M = \chi(L^{-1})_M$ and $\chi(M) = T(M)$, then $\chi(M,L) = T(M,L)$. □

THEOREM 4.2 (Riemann Roch Theorem for Vector Bundles). *Let M be an algebraic manifold of dimension n and let E be a holomorphic vector bundles over M with fiber $\mathbb{C}_q$. Let $c_0, c_1, \ldots, c_n$ be the Chern classes of M and $d_0, d_1, \ldots, d_q$ the Chern classes of M with $c_0 = d_0 = 1; c_i, d_i \in H^{2i}(M, \mathbb{Z})$. The cohomolology groups $H^i(M, E)$ are finite dimensional vector spaces which vanishes for $i \geq n$. The Euler characteristic*

$$\chi(M, E) = \sum_{i=0}^{n} (-1)^i dim H^i(M, E)$$

can be expressed as a "polynomial" $T(M, E)$ in the Chern classes c_i and d_i:

$$\begin{aligned} \chi(M, E) &= \varkappa_n \left[(e^{\delta_1} + \cdots + e^{\delta_q}) \Pi_{i=1}^{i=n} \frac{\gamma_i}{1 - e^{-\gamma_i}} \right] \\ &= \varkappa_n \left[e^{c_1/2} (e^{\delta_1} + \cdots + e^{\delta_q}) \Pi_{i=1}^{n} \frac{\gamma_i/2}{\sinh \gamma_i/2} \right] = T(M, E). \end{aligned}$$

where

$$\sum_{i=0}^{n} c_i x^i = \Pi_{i=1}^{n} (1 + \gamma_i x) \text{ and } \sum_{i=0}^{q} d_i x^i = \Pi_{i=1}^{q} (1 + \delta_i x)$$

and the term of degree n of the expression in square brackets is a polynomial in the c_i and d_i. This term determines an element of $H^{2n}(M, \mathbb{Z}) \otimes \mathbb{Q}$ which is to be evaluated on the fundamental $2n$-dimensional cycle of M.

PROOF. Let P be the principal bundle associated the the $GL(q, \mathbb{C})$-bundle ξ associated to $E \longrightarrow M$ define $\rho : P/\Delta(q, \mathbb{C}) \longrightarrow M$. By Theorem 3.3, we have $T(M, E) = T(P/\Delta(q, \mathbb{C}), \rho^* E)$. Further, by the theorem of Borel proved by using spectral sequences in [Hir] Appendix II, we have $\chi(M, E) = \chi(P/\Delta(q, \mathbb{C}), \rho^* E) \chi(F(q))$. Together with $\chi(F(q)) = 1$, we get $\chi(M, E) = \chi(P/\Delta(q, \mathbb{C}), \rho^* E)$. Since $P/\Delta(q, \mathbb{C})$ is a split manifold, let $A_1, \ldots, A_q$ be the q diagonal $\mathbb{C}^*$-bundles associated to ξ^Δ, that is $\rho^* \xi^\Delta = (A_1, \ldots, A_q)$. By the property of T in Theorem 2.1 in Chapter 5 and the property of χ, we have,

$$T(P/\Delta(q, \mathbb{C}), \rho^* E) = T(P/\Delta(q, \mathbb{C}), \rho^* \xi^\Delta) = \sum_{i=1}^{q} T(E, A_i);$$

$$\chi(P/\Delta(q, \mathbb{C}), \rho^* E) = \chi(P/\Delta(q, \mathbb{C}), \rho^* \xi^\Delta) = \sum_{i=1}^{q} \chi(P/\Delta(q, \mathbb{C}), A_i),$$

respectively. By the Riemann-Roch theorem for line bundles, we have $\chi(P/\Delta(q, \mathbb{C}), A_i) = T(P/\Delta(q, \mathbb{C}), A_i)$, for $i = 1, \ldots, q$. Hence $\chi(M, E) = T(M, E)$. □

REMARK 4.4. [Hir](Appendix I) also provide the proof of the above Riemann-Roch formula of vector bundles over a compact complex manifold instead of an algebraic manifold. Our calculation in the next chapter will use the former one.

CHAPTER 7

Deformations of Complex Structures

The idea of deformation goes back to Riemann. He considered the deformations of complex structures of complex curves and gave the number of effective parameters of the deformation in terms of genus, which is a topological invariant of curves. This shows that genus is a topological invariant which can give the freedom of deformation of a curve. The greater the genus is, the more freedom the curve can deform. Kodaira and Spencer invented the theory of the deformation of complex structures of compact complex manifolds. We will sketch their idea and list without proof a number of fundamental theorems they have obtained in [Kodaira1] and [Kodaira2]. We will also state Kuranishi's generalization on their deformation theory in the local sense in [Kuranishi]. As an application of both the deformation theory of Kuranishi and the Riemann-Roch theorem, we will bound the freedom of deformation locally by polynomials of Chern classes and make some observation. The topic and the two parts of calculation of Euler-Poincaré characteristics by the Riemann-Roch theorem are suggested by Professor Ngaiming Mok.

1. Deformations of Complex Structures of Compact Complex Manifolds

This section sketches Kodaira and Spencer's idea of deformation of complex structures. We will first give a concrete model of deformation of complex structures of compact complex manifolds parameterized by a complex manifold and the Kodaira-Spencer map, which realizes the parameterization. Then we study the moduli space of all complex structures of a compact complex manifold. In particular, we will give the definition of the number of moduli which measures the freedom of deformation of a compact complex manifold and whose definition relies on the existence of certain complete and effective family of manifolds. After this, we will state Kodaira and Spencer's result about the existence of such family under the condition that $H^2(M,\Theta)$ vanishes where Θ is the sheaf of germs of holomorphic sections of the tangent bundle TM. Finally, we will give Kuranishi's theory as a generalization of the existence theorem of Kodaira and Spencer in the local sense by releasing the condition $H^2(M,\Theta)=0$ which may not be satisfied in general.

1.1. Deformation of Complex Structures of Compact Complex Manifolds Parameterized by a Complex Manifold. This subsection gives the model of Kodaira and Spencer concerning the deformation of complex structures of compact complex manifolds parameterized by a complex manifold. The main references are [Kodaira2] and [Shimizu].

The following definition is from [Shimizu].

DEFINITION 1.1 (Complex Analytic Family of Compact Complex Manifolds over a Complex Manifold). A holomorphic mapping $\pi : \mathcal{V} \longrightarrow W$ from an $(m+n)$-dimensional complex manifold $\mathcal{V}$ to an m-dimensional complex manifold $\mathcal{W}$ satisfying the following conditions is called a *complex analytic family of compact complex manifolds* over the complex manifold $\mathcal{W}$.

(i) π is a proper mapping. That is for any compact set K of the complex manifold $\mathcal{W}$, the inverse image $\pi^{-1}(K)$ is compact.
(ii) π is a smooth holomorphic mapping. That is, for any point $P \in \mathcal{V}$ the linear mapping $(d\pi)_P : T_P(\mathcal{V}) \longrightarrow T_{\pi(P)}(\mathcal{W})$ of the holomorphic tangent spaces is surjective.
(iii) For any point $w \in \mathcal{W}$, the fiber $\pi^{-1}(\mathcal{W})$ of each point $w \in \mathcal{W}$ is a compact complex manifold.

For a fixed $w_0 \in \mathcal{W}$ defined above, and w be any point in $\mathcal{W}$, the manifold $\mathcal{V}_w$ is a *deformation* of the manifold $\mathcal{V}_{w_0}$.

Choose a coordinate neighborhood U of the point w_0 in $\mathcal{W}$. Denote $w \in U$ by the coordinates $(w^1, \ldots, w^m)$. In particular, denote w_0 by the coordinates $(w_0^1, \ldots, w_0^m)$. Cover the inverse image $\pi^{-1}(U) \subset \mathcal{V}$ by open sets $\{U_\lambda\}_{\lambda\in\Lambda}$. For each U_λ, choose the local coordinates $(z_\lambda^1, \ldots, z_\lambda^n, w^1, \ldots, w^m)$ with transition functions $f^i_{\lambda\mu}$ given by

$$z^i_\lambda = f^i_{\lambda\mu}(z^1_\mu, \ldots, z^n_\mu, w^1, \ldots, w^m), \quad i = 1, 2, \ldots, n.$$

The portion of the first n-coordinates of U_λ is like a tube lying in $\mathcal{V}$.

When restricted to a certain fiber, say $\mathcal{V}_{w_0}$, the cross section of each tube in the manifold becomes an open covering of the manifold $\mathcal{V}_{w_0}$. Let $q \in U_\lambda \cap U_\mu, \lambda, \mu \in \Lambda$, such that

$$q = \begin{cases} ({z_\lambda}^1, \ldots, {z_\lambda}^n, {w_0}^1, \ldots, {w_0}^m) & \text{as a point in the open set } U_\lambda \cap \mathcal{V}_{w_0} \\ ({z_\mu}^1, \ldots, {z_\mu}^n, {w_0}^1, \ldots, {w_0}^m) & \text{as a point in the open set } U_\mu \cap \mathcal{V}_{w_0} \end{cases}$$

and the transition function is

$${z_\lambda}^i = f^i_{\lambda\mu}({z_\mu}^1, \ldots, {z_\mu}^n, w_0^1, \ldots, w_0^m), \quad 1 \leq i \leq n.$$

The directional derivative with respect to $v_{w_0 \longrightarrow w}$,

$$\sum_{j=1}^{m} \frac{\partial}{\partial w^j} f^i_{\lambda\mu}({z_\mu}^1, \ldots, {z_\mu}^n, w_0^1, \ldots, w_0^m), \quad 1 \leq i \leq n,$$

is called the *infinitesimal deformation* of the complex manifold $\mathcal{V}_{w_0}$.

Define the tangent vector

$$\theta^{(j)}_{\lambda\mu} = \sum_{i=1}^{n} \frac{Df^i_{\lambda\mu}}{dw^j}({z_\mu}^1, \ldots, {z_\mu}^n, {w_0}^1, \ldots, {w_0}^m)\frac{\partial}{\partial {z_\mu}^i}, \quad 1 \leq j \leq m$$

where $\frac{D}{Dw^j}$ is the projection of the directional derivative $\frac{\partial}{\partial w^j}$ on the the tangent space along fiber. For each fixed $j = 1, \ldots, m$, $\theta^{(j)}_{\lambda\mu}$ can be considered as a 1-cochain in the 1-cochain group $C^1(\mathcal{U}|\mathcal{V}_{w_0}, \Theta_{\mathcal{V}/\mathcal{W}})$ where $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ and $\Theta_{\mathcal{V}/\mathcal{W}}$ is the sheaf of germs of holomorphic vector fields on along the fibers.

Proposition 1.1. *The* 1*-cochain* $\{\theta^{(j)}_{\lambda\mu}\}, \quad 1 \leq j \leq m$ *is a cocycle.*

Proof. The direct computation can be found in [Shimizu] p. 5. □

Thus, $\{\theta^{(j)}_{\lambda\mu}\}$ defines a class in the Čech cohomology group $H^1(\mathcal{U}|\mathcal{V}_{w_0}, \Theta_{\mathcal{V}/\mathcal{W}})$. Taking direct limit, we get $\check{\mathrm{H}}^1(\mathcal{V}_{w_0}, \Theta_{\mathcal{W}/\mathcal{V}})$. We denote the class also by $\{\theta^{(j)}_{\lambda\mu}\}$ where $1 \leq j \leq m$.

Definition 1.2 (Kodaira-Spencer Map). The map

$$\rho : T_{w_0}\mathcal{W} \longrightarrow H^1(\mathcal{V}_{w_0}, \Theta_{\mathcal{V}/\mathcal{W}})$$

defined by

$$\rho : \sum_{j=1}^{m} a_j \frac{\partial}{\partial w^j} \mapsto \sum_{j=1}^{m} a_j \{\theta^{(j)}_{\lambda\mu}\}$$

is called the *Kodaira-Spencer Mapping.*

As stated in [Shimizu] p. 5, the cohomology class $\{\theta^{(j)}_{\lambda\mu}\}$ is uniquely determined if the coordinates $\{w^1, \ldots, w^m\}$ is fixed and the class is independent of the choice of open covering $\mathcal{U}$. Note that $\rho(\sum_{j=1}^m a_j \frac{\partial}{\partial w^j})$ will map a vector v in the tangent space of $\mathcal{V}_{w_0}$ to the projection of its directional derivative with respect to $\sum_{j=1}^m a_j \frac{\partial}{\partial w^j}$ from the $(n+m)$-dimensional tangent space to the tangent space along the fiber.

1.2. Number of Moduli and Theorem of Existence. Consider the set of all equivalence classes of complex structures on an underlying even-dimensional smooth manifold. We call this the moduli space of complex structures. If the moduli space can be given the structure of a complex analytic space, for instance a complex manifold, in which case it is locally a domain in a complex space, then we may use its dimension to measure the freedom of deformations of the compact complex manifold. Kodaira and Spencer call this dimension "the number of parameters". They were able to calculate this number even before rigorously defining it on several simple examples like the torus and projective space, see [Kodaira2]. To get a rigorous definition of the "number of parameter", they use a good domain in a vector space to parameterize the deformation of complex structures of a compact complex manifold. It is good in the sense that it is both large enough to parameterize all local deformations and it is small enough to get injectivity of the parameterization.

After showing the independence of the choices of such parameterization domain, the number of parameter is defined by the dimension of the parameterization domain, which is known as the *number of moduli*.

Definition 1.3 (Effectiveness). Let $(\mathcal{M}, B, \pi)$ be a complex analytic family of compact complex manifolds where B is a domain in $\mathbb{C}^m$. If for $t \in B$ the Kodaira-Spencer map

$$\rho_t : T_t(B) \longrightarrow H^1(M_t, \Theta_t)$$

is injective, then we say that the family $(\mathcal{M}, B, \pi)$ is *effective* at $t \in B$.

If the family $(\mathcal{M}, B, \pi)$ is effective at every $t \in B$, then $(\mathcal{M}, B, \pi)$ is called an *effectively parameterized complex analytic family.*

Before giving the definition of completeness we will first give the method of inducing a complex analytic family from another. Suppose $(\mathcal{M}, B, \pi)$ is a complex analytic family of compact complex manifolds, where B is a domain in $\mathbb{C}^n$. Let D be a domain in $\mathbb{C}^r$, and

$$h : D \longrightarrow B,$$

which maps t to $s = h(t)$ for any $t \in D$, is a holomorphic map. Then we can induce a new complex analytic family $(\mathcal{N}, D, \omega)$ by changing the parameter from t to s as follows:

Define the holomorphic map

$$\Pi : \mathcal{M} \times D \longrightarrow B \times D$$

by $\Pi(p, s) = (t, s) = (\pi(p), s)$. Hence $(\mathcal{M} \times D, B \times D, \Pi)$ is a complex analytic family. The graph of h

$$G = \{(h(s), s) \in B \times D | s \in D\}$$

is a submanifold of $B \times D$ and hence $\mathcal{N}$ defined by $\Pi^{-1}(G)$ is a submanifold of the complex manifold $\mathcal{M} \times D$, hence $(\mathcal{N}, G, \Pi)$ where Π here means $\Pi|_{\mathcal{N}}$ is a complex analytic family. Further, since the projection

$$P : B \times D \longrightarrow D$$

maps G biholomorphically onto D, that is $G = P(D)$. Further define $\omega : \mathcal{N} \longrightarrow D$ by $\omega = P \circ \Pi$, then $(\mathcal{N}, D, \omega)$ is a complex analytic family. It is called the *complex analytic family induced from* $(\mathcal{M}, B, \pi)$ *by the holomorphic map* $h : D \longrightarrow B$.

Definition 1.4 (Completeness). Let $(\mathcal{M}, B, \pi)$ be a complex analytic family of compact complex manifolds. If for $t_0 \in B$, and for any complex analytic family $(\mathcal{N}, D, \omega)$ such that D is a domain in $\mathbb{C}^l$ containing 0 and that $\omega^{-1}(0) = \pi^{-1}(t_0)$, there exists a sufficiently small domain E with $0 \in E \subset D$, and a holomorphic map $h : s \longrightarrow t$ with $h(0) = t_0$ such that $(\mathcal{N}_E, E, \omega)$ is the complex analytic family induced form $(\mathcal{M}, B, \pi)$ by h where $\mathcal{N}_E = \omega^{-1}(E)$. Then we say that $(\mathcal{M}, B, \pi)$ is *complete at* t_0.

If the family $(\mathcal{M}, B, \pi)$ is complete at every point of B, then $(\mathcal{M}, B, \pi)$ is called a *complete complex analytic family.*

Note that if $(\mathcal{M}, B, \pi)$ is complete at one point t_0, then the family contains all sufficiently small deformations of M.

Definition 1.5 (Number of Moduli). Let M be a compact complex manifold. If there exists an effectively parameterized and complete complex analytic family $(\mathcal{M}, B, \pi)$ with $\pi^{-1}(0) = M$ where B is a domain in $\mathbb{C}^m$ containing 0, then $m(M)$ defined by $m = dimB$ is called the *number of moduli* of M.

Note that the number of moduli which measures the freedom of all possible deformations of a compact complex manifold is globally defined at each point of the compact complex manifold. The *number of parameter* of M is then defined to be the number of moduli. If there is no such family then we say that *the number of moduli is not defined* and *the number of parameter cannot be determined*.

To know the number of moduli for a compact complex manifold M, it suffices to find an effective parameterized and complete analytic family by the definition of the number of moduli. The question reduces to the existence of such a family of a given compact complex manifold. Kodaira and Spencer find out that under the condition of $H^2(M, \Theta)$ vanishing, if such family exists, then the number of moduli $m(M)$ of M will be equal to $dimH^1(M, \Theta)$. Further they give a sufficient condition for the existence of such effectively parameterized and complete complex analytic family in [Kodaira2]. We summarize their results in the following theorem.

Theorem 1.1. *Suppose $H^2(M, \Theta) = 0$. Then $(\mathcal{M}_\Delta, \Delta, \pi)$ where Δ is a poly-disk in $\mathbb{C}^m$ sufficiently small, is an effectively parameterized and complete family and hence the number of moduli $m(M)$ is defined and*

$$m(M) = dimH^1(M, \Theta), \tag{86}$$

if and only if $dimH^1(M_t, \theta_t)$ is independent of $t \in \Delta$.

1.3. Kuranishi's Theory. There is a generalization of these results in a local sense by further releasing the condition $H^2(M, \Theta) = 0$. This is Kuranishi's theory. Instead of considering B as a domain in $\mathbb{C}^m$, Kuranishi's B becomes an analytic subset. In other words, B is locally the zero set in $\mathbb{C}^m$ of a finite number of holomorphic functions. His results, Theorem 2 and Theorem 3 in [Kuranishi], are summarized as follows:

Theorem 1.2. *For any compact complex manifold M, there exists a complete complex analytic family $(\mathcal{M}, B, \pi)$ with $0 \in B \subset \Delta(\varepsilon)$ and $\pi^{-1}(0) = M$, which is called the Kuranishi family. Furthermore, the Kuanishi family is effective at $0 \in B$.*

The conclusion in Theorem 1.2 is weaker than that in Theorem 1.1 in the sense that it only gives the effectiveness at $0 \in B$. Theorem 1.2 is more general than Theorem 1.1 in the sense that it relaxes the condition of $H^2(M, \Theta) = 0$.

Theorem 1.2 gives bound on the freedom of deformation of M locally. Indeed, by Kuranishi's theory in [Kuranishi], $dim\Delta(\varepsilon)$ is shown to be equal to

$dimH^1(M,\Theta)$. Further, let $l = dimH^2(M,\Theta)$, then B is constructed as an analytic subset defined by l holomorphic functions $f_1(t) = \cdots = f_l(t) = 0$. Hence

$$dimB \geq h^1(M,\Theta) - h^2(M,\Theta), \tag{87}$$

where $h^i(M,\Theta) = dimH^i(M,\Theta)$. The proof of inequality involves techniques from analysis, including Hölder norm and Newlander-Nirenberg's theorem (Theorem 5.5 in [Kodaira2]), which I am not able to give details. We only outline the main steps in the proof (87) by Kuranishi. The reference is [Kodaira2] p. 315.:

PROOF. Step 1. The integrability condition in the form of Dolbeault cohomology is

$$\bar{\varphi}(t) + \frac{1}{2}[\varphi(t), \varphi(t)] = 0.$$

In other words, given a reference compact complex manifold M, if its deformation is parameterized by some polydisk Δ and the fiber above $0 \in \Delta$ is M, and if $\varphi(t)$ is a $(0,1)$-form on M which gives deformation then it must satisfy the integrability condition $\bar{\varphi}(t) = \frac{1}{2}[\varphi(t), \varphi(t)]$ where $[\quad,\quad]$ is the Lie bracket.

Step 2. $H[\varphi(t), \varphi(t)] = 0$ if and only if the integrability condition. Due to harmonic theory for any the group of $(0,q)$-forms of M is the direct sum

$$A^{0,q}(M) = \mathbb{H}^{0,q}(M) \oplus \Box A^{0,q}(M)$$

where $B^{0,q}(M)$ is the group of harmonic $(0,q)$-forms of M and $\Box$ is the complex Beltrami-Laplace operator. H above is the linear operator send $A^{0,q}(M)$ to the component $\mathbb{H}^{0,q}(M)$ by projection. This is the stage where Hölder norm and the *a priori inequality* is used.

Step 3. $dimB$ =number of moduli of deformation of M. Write $H[\varphi(t), \varphi(t)] = \sum_{k=1}^{l} f_k(t)\gamma_k$ where $\{\gamma_1, \ldots, \gamma_l\}$ is an orthonormal basis of $\mathbb{H}^{0,2}(M)$ and the coefficients $f_k(t) = ([\varphi(t), \varphi(t)], \gamma_k), k = 1, \ldots, l$ are holomorphic functions of t. Hence $\varphi(t)$ satisfies the integrability condition if and only if $f_1(t) = \cdots = f_l(t) = 0$. Set B be the analytic subset defined by

$$B = \{t \in \Delta(\varepsilon) | f_1(t) = \cdots = f_l(t) = 0\}.$$

Then the parameterization given by the subset B of deformation of M gives all possible deformations of M. On the other hand, by Newlander-Nirenberg theorem [Kodaira2] p. 268, for ε small enough, each $t \in \Delta(\varepsilon)$ and hence each $t \in B \subset \Delta(\varepsilon)$ will define a complex structure on the compact manifold M. Hence $dimB = m_0(M)$ where $m_0(M)$ is the number of moduli of deformation of M.

Step 4. $m_0(M) \geq h^1(M,\Theta) - h^2(M,\Theta)$. Indeed, $m_0(M) = dimB$ and $dimB = dim\mathbb{H}^{0,1}(M) -$number of linearly independent functions in $\{f_1(t), \ldots, f_l(t)\}$, which is less than or eqaul to $dim\mathbb{H}^{0,2}(M)$. Hence $m_0(M) \geq h^1(M,\Theta) - h^2(M,\Theta)$ since $dimH^q(M,\Theta) = dim\mathbb{H}^{0,q}(M)$ for all $q \geq 0$. $\Box$

1.4. Calculation of $\chi(M,\Theta)$ and Observations on Deformations. Suppose M is a compact complex manifold of complex dimension 2, locally the freedom of deformation will be given by (87). Since all $h^i(M,\Theta)$ vanishes for $i>2$, then the Euler characteristic will be

$$\chi(M,\Theta)=h^0(M,\Theta)-h^1(M,\Theta)+h^2(M,\Theta).$$

Since $h^0(M,\Theta)\geq 0$, we have

$$m_0(M)\geq -\chi(M,\Theta)+h^0(M,\Theta)\geq -\chi(M,\Theta). \tag{88}$$

Let us compute $\chi(M,\Theta)$ by using the Riemann-Roch theorem. Suppose M is of Chern classes $c_0=1$, c_1 and c_2 with formal factorization

$$1+c_1x+c_2x^2=(1+\gamma_1x)(1+\gamma_2x),$$

then

$$\begin{aligned}
&\chi(M,\Theta)\\
=\ &\varkappa_2\left[e^{c_1/2}\frac{\gamma_1/2}{\sinh\gamma_1/2}\frac{\gamma_2/2}{\sinh\gamma_2/2}(e^{\gamma_1}+e^{\gamma_2})\right]\\
=\ &\varkappa_2\left[(1+\frac{1}{2}c_1+\frac{1}{8}c_1^2)(1-\frac{1}{24}\gamma_1^2)(1-\frac{1}{24}\gamma_2^2)(1+\gamma_1+\frac{1}{2}\gamma_1^2+1+\gamma_2+\frac{1}{2}\gamma_2^2)\right]\\
=\ &\varkappa_2\left[(1+\frac{1}{2}c_1+\frac{1}{8}c_1^2)(1-\frac{1}{24}c_1^2+\frac{1}{12}c_2)(2+c_1+\frac{1}{2}c_1^2-c_2)\right]\\
=\ &\varkappa_2\left[(1-\frac{1}{24}c_1^2+\frac{1}{12}c_2+\frac{1}{2}c_1+\frac{1}{8}c_1^2)(2+c_1+\frac{1}{2}c_1^2-c_2)\right]\\
=\ &\varkappa_2\left[(1+\frac{1}{2}c_1+\frac{1}{12}c_1^2+\frac{1}{12}c_2)(2+c_1+\frac{1}{2}c_1^2-c_2)\right]\\
=\ &\frac{1}{2}c_1^2-c_2+\frac{1}{2}c_1^2+\frac{1}{6}c_1^2+\frac{1}{6}c_2\\
=\ &\frac{1}{6}(7c_1^2-5c_2)
\end{aligned}$$

Hence by the inequality (88), we have the freedom of deformation of M locally is given by

$$m_0(M)\geq -\frac{1}{6}(7c_1^2-5c_2). \tag{89}$$

Note that when $c_1^2\geq\frac{5}{7}c_2$ we get no conclusion of deformations of M locally and when $c_1^2<\frac{5}{7}c_2$, M must have deformations.

2. Deformations of Complex Structures of Vector Bundles

We will first study the generalized theory of deformations of complex structures of vector bundles parameterized by a complex manifold as a concrete image of Kodaira and Spencer in [Kodaira1]. Secondly, we will study the generalized theory of Kuranishi's theory on the local freedom of deformations of complex structures of Hermitian vector bundles with fixed base manifolds. Finally, this section will also discuss the case when the deformations of vector bundles of fixed determinant line bundle. That is, the deformations of vector bundles that are not simply obtained from tensoring with a family of line bundles, which is not all trivial.

2.1. Deformations of Complex Structures of Vector Bundles over a Complex Manifold. Here we use the language of algebraic topology in [Kodaira1] compared with the subsection 1.1 which use the language of differential geometry.

Definition 2.1 (Complex Analytic Family of Complex Fiber Bundles). Let $\mathcal{V} \longrightarrow \mathcal{W}$ be a family of compact complex manifolds. If $\mathcal{B} \longrightarrow \mathcal{V}$ is a complex fiber bundle with structure group a complex Lie group G. If the restriction of $\mathcal{B}$ on any fiber of $\mathcal{V}$ is still a complex fiber bundle over that fiber as a base manifold, then we call $\mathcal{B} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}$ a *complex analytic family of complex fiber bundles over* $\mathcal{W}$.

In particular if both $\mathcal{B} \longrightarrow \mathcal{V}$ and $\mathcal{B} \longrightarrow \mathcal{W}$ are complex holomorphic vector bundles. Then $\mathcal{B} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}$ is called a *complex analytic family of holomorphic vector bundles over* $\mathcal{W}$.

Let $\mathcal{P} \longrightarrow \mathcal{V}$ be the principal fiber bundle associated to a complex vector bundle $\mathcal{B} \longrightarrow \mathcal{V}$. To study the infinitesimal deformation of the complex analytic family of complex holomorphic vector bundles $\mathcal{B} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}$, it suffices to study the infinitesimal deformation of the associated family of complex principal fiber bundles $\mathcal{P} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}$.

Definition 2.2 (Fundamental Sequence of Vector Bundles). Let $\mathcal{V} \longrightarrow M$ be a complex analytic family of holomorphic vector bundles. Let $\mathfrak{g} \longrightarrow \mathcal{V}$ be the tangent bundle of $\mathcal{V}$ and $\mathfrak{f} \longrightarrow \mathcal{V}$ be the subbundle of $\mathfrak{g}$ consisting of all tangent vector of $\mathcal{V}$ which is tangent to the fiber of $\mathcal{V} \longrightarrow M$. Then the exact sequence of vector bundles over $\mathcal{V}$:

$$0 \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{f} \longrightarrow 0 \tag{90}$$

is called the *fundamental sequence of vector bundles for the family* $\mathcal{V} \longrightarrow M$.

The family $\mathcal{P} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}$ can be regarded in different ways: (i)$\mathcal{P} \longrightarrow \mathcal{W}$. (ii)$\mathcal{P} \longrightarrow \mathcal{V}$. For (i), let

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathcal{F} \longrightarrow$$

be the fundamental sequence of vector bundles (90) of the family $\mathcal{P} \longrightarrow \mathcal{W}$. Let $\mathfrak{f}$ be the subbundle of $\mathcal{F} \longrightarrow \mathcal{P}$ consisting of tangent vectors which is tangent to the fibers of $\mathcal{P} \longrightarrow \mathcal{V}$ and call $\mathfrak{f}$ the bundle along fibers of the family $\mathcal{P} \longrightarrow \mathcal{V}$. We thus get an exact sequence of vector bundles over $\mathcal{P}$:

$$0 \longrightarrow \mathfrak{f} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{f} \longrightarrow 0.$$

From this we may define

$$\mathfrak{d} = \mathfrak{f}/G, \quad \mathfrak{w} = \mathcal{F}/G, \quad \mathfrak{r} = \mathfrak{g}/G$$

by identifying elements in fibers up to an operation from the structure group G of $\mathcal{P}$. Hence we get the following exact commutative diagram of vector bundles:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \mathfrak{f} & \to & \mathfrak{g} & \to & \mathfrak{g}/\mathfrak{f} & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \mathfrak{w} & \to & \mathfrak{r} & \to & \mathfrak{r}/\mathfrak{w} & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \mathfrak{d} & \to & \mathfrak{d} & \to & 0 & & \\ & & \uparrow & & \uparrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

which is called the *fundamental bundle diagram for the family* $\mathcal{P} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}$. We can now define the corresponding sheaf of holomorphic germs diagram in a canonical way, for details refer to [Kodaira1] p. 354.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \Theta & \to & \Pi & \to & T & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \Sigma & \to & \Gamma & \to & T & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \Xi & \to & \Xi & \to & 0 & & \\ & & \uparrow & & \uparrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

which is called the *fundamental sheaf diagram for the family* $\mathcal{P} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}$. The corresponding long exact sequence of the third line is

$$0 \longrightarrow H^0(\mathcal{W}, \Sigma) \longrightarrow H^0(\mathcal{W}, \Gamma) \longrightarrow H^0(\mathcal{W}, T) \xrightarrow{\delta^*} H^1(\mathcal{W}, \Sigma) \longrightarrow \cdots$$

will give the map

$$\delta^* : H^0(\mathcal{W}, T) \longrightarrow H^1(\mathcal{W}, \Sigma),$$

which is the *Kodaira-Spencer map* of the family of vector bundles $\mathcal{B} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}$.

2.2. Deformations of Complex Structures of Vector Bundles with Fixed Base Manifolds and the Corresponding Kuranishi's Theory. From now on we narrow down our consideration to a specific case. Namely, the deformation of complex structures of vector bundles with fixed base manifolds. That is, the base manifold will not deform. Further as an analogue of Kuranishi's theory for deformations of compact complex manifolds, in particular the inequality (87), we will give Kuranishi's theory for deformations of complex structures of vector bundles in the case when the base manifold does not deform.

Let $V \xrightarrow{\pi} M$ be the reference vector bundle of rank r. Let $\{U_i\}_{i\in I}$ be an open covering of M, then $\{V_i\}_{i\in I}$ defined by $V_i = \pi^{-1}(U_i), i \in I$ is an open covering of the vector bundle $V \longrightarrow M$. Suppose $h_i : V_i \longrightarrow U_i \times \mathbb{C}^r$ is the trivialization of V and the corresponding transition functions are

$$g_{ij} : U_{ij} \longrightarrow GL(r, \mathbb{C})$$

with $(x, g_{ij}(x)(v)) = h_j \circ h_i^{-1}(x, v)$ where $x \in U_{ij}$ and v is a vector in $\mathbb{C}^r$ of $U_i \times \mathbb{C}^r$. Let $\Delta(\varepsilon)$, the polydisk centered at the origin and of radius ε in $\mathbb{C}^m$, be the parameterization space of deformation of V and the family of vector bundles denoted as $\{\mathcal{W}, \Delta(\varepsilon), \omega\}$ where $\omega^{-1}(0) = V$. Note that in this case deformation of V is equivalent to deformation of $\{h_i\}$. That is $\{h_i(0)(x) = h_i(x)\}$ for any $x \in M$ can deform with respect to $t \in \Delta(\varepsilon)$ to $\{h_i(t)(x)\}$ provided the corresponding transition functions $\{g_{ij}(t)(x)\}$ again satisfies cocycle relation. This is equivalent to say that $\{g_{ij}(t)\}$ will be cocycle for any $t \in \Delta(\varepsilon)$ and defines a class $\{g_{ij}(t)\} \in H^1(M, EndV)$. As an analogue of Kuranishi's theory for deformation of complex structures of compact complex manifolds, in particular the inequality (87), we will have the inequality:

$$m_0(V) \geq h^1(M, EndV) - h^2(M, End_0V). \tag{91}$$

where End_0V is the trace-free part of $EndV$. The reference is [Fukaya] p. 142. and [F-M] p. 302. I can only outline the proof:

Proof. Step 1. The integrability condition in the form of Dolbeault cohomology in this case is

$$\bar{\partial}A(t) + \frac{1}{2}[A(t), A(t)] = 0.$$

That is, given a reference vector bundle V, if its deformation is parameterized by some polydisk $\Delta(\varepsilon)$ and the fiber above $0 \in \Delta$ is V, then $A(t) \in A^{0,1}(M)$ determines a deformation of V if and only if $A(t)$ satisfies this integrability condition for sufficiently small ε.

Step 2. $H[A(t), A(t)] = 0$ if and only if the integrability condition. Due to harmonic theory for any the group $A^{0,q}(M, EndV)$ is the direct sum

$$A^{0,q}(M, EndV) = \mathbb{H}^{0,q}(M, EndV) \oplus \bar{\partial}(A^0(M, EndV)) \oplus \bar{\partial}^*(A^{0,2}(M, EndV)).$$

H above is the linear operator send $A^{0,q}(M, EndV)$ to the component $\mathbb{H}^{0,q}(M, EndV)$ by projection.

Step 3. $dimB$ =number of moduli of deformation of M. It can be shown further that the image of H is $\mathbb{H}^{0,2}(M, End_0V)$ instead of $\mathbb{H}^{0,2}(M, EndV)$ where $End_0(V)$ is the trace free subset of $EndV$. Write $H[A(t), A(t)] = \sum_{k=1}^{l} f_k(t)\gamma_k$ where $\{\gamma_1, \ldots, \gamma_l\}$ is an orthonormal basis of $\mathbb{H}^{0,2}(M, End_0V)$ and the coefficients $f_k(t) = ([A(t), A(t)], \gamma_k), k = 1, \ldots, l$ are holomorphic functions of t. Hence $A(t)$ satisfies the integrability condition if and only if $f_1(t) = \cdots = f_l(t) = 0$. Set B be the analytic set defined by

$$B = \{t \in \Delta(\varepsilon) | f_1(t) = \cdots = f_l(t) = 0\}.$$

Then the parameterization given by the subset B is gives all and deformation of M gives all possible deformations of M and each point $t \in B$ will give a deformation of V. Hence $dimB = m_0(V)$ where $m_0(V)$ is the number of moduli of deformation of V.

Step 4. $m_0(V) \geq h^1(M, EndV) - h^2(M, End_0V)$. From $m_0(V) = dimB$ and $dimB = dim\mathbb{H}^{0,1}(M) -$number of linearly independent functions in $\{f_1(t), \ldots, f_l(t)\}$, which is less than or equals to $dim\mathbb{H}^{0,2}(M, End_0V)$. Hence $m_0(V) \geq h^1(M, EndV) - h^2(M, End_0V)$ since $dimH^q(M, X) = dim\mathbb{H}^{0,q}(M, X)$ for all vector bundle X over M and $q \geq 0$. □

2.3. Deformations of Vector Bundles with Fixed Determinant Line Bundles. This subsection studies in particular the deformations of vector bundles with fixed base manifolds and fixed determinant line bundles, since as we will explain later such kind of deformations rules out the deformations given by simply tensoring with a family of not all trivial line bundles to the original vector bundle.

DEFINITION 2.3 (Determinant Line Bundle). Let $V^r \longrightarrow M$ be a vector bundle with frames $\{e_1, \ldots, e_r\}$ and transition functions $\{g_{ij}\}$ with respect to an open covering $\{U_i\}_{i\in I}$ of M. The *determinant line bundle corresponding to* V, $det(V) \longrightarrow M$ is a line bundle with transition functions $\{det(g_{ij})\}$ and frames given by $e_1 \wedge \cdots \wedge e_r$.

Suppose we have a vector bundle $V \longrightarrow M$ with basis $\{e_1, \ldots, e_n\}$ and transition function $\{g_{ij}\}$ and a line bundle $L \longrightarrow M$ with basis $\{f\}$ and transition function $\{h_{ij}\}$ where $h_{ij}(x) \in \mathbb{C}^*$ for any $x \in U_i \cap U_j$. Then the tensor bundle of V and L

$$V \otimes L \longrightarrow M$$

is a vector bundle of rank r, of basis $\{e_1 \otimes f, \ldots, e_r \otimes f\}$ and transition functions $\{g_{ij} \otimes h_{ij}\}$. The corresponding determinant line bundle of $V \otimes L$ is thus

$$det(V \otimes L) \longrightarrow M$$

with basis $\{(e_1 \otimes f) \wedge \cdots \wedge (e_r \otimes f)\}$ and transition functions $det(g_{ij} \otimes h_{ij}) = det(g_{ij}) \cdot det(h_{ij})$. Therefore, if a deformation of a vector bundle is constructed by purely tensoring a family of line bundles in which not all are trivial line bundles, then the determinant above cannot be constant. Therefore to rule out the deformations caused purely by tensoring with a family of not all trivial line bundles, we have to consider the deformations of a vector bundle whose determinant line bundle has fixed determinant. It can be shown that this is equivalent to study the deformations of vector bundle $End_0V \longrightarrow M$ where $End_0(V)$ denotes the subbundle of $End(V)$ such that it is of trace zero whose elements are endomorphisms that change direction of vectors in V.

As an analogue to the inequality (91). We have locally the bound of freedom of deformations of V as follows,

$$m_0(V) \geq h^1(M, End_0V) - h^2(M, End_0V). \tag{92}$$

where End_0V stands for sheaf of germs of holomorphic sections of the vector bundle End_0V. The proof of this inequality can be found in [F-M] p. 300.

3. Deformations of Tangent Bundle as a Vector Bundle

Section 1 has given the deformation theory of complex structures of a compact complex manifold. Section 2 has given the deformation theory of complex structures of vector bundles over a fixed base manifold. In general, starting from a complex manifold M with tangent bundle TM it can be deformed in the following procedure: Firstly, let the complex structure M deform and end in M', hence the resulting corresponding tangent bundle will deform to TM'. Secondly, start from the tangent bundle $TM' \longrightarrow M'$ and fix the base manifold M', we may deform TM' as a vector bundle to, say $E \longrightarrow M'$. Then the composition of the two steps gives a general deformation from $TM \longrightarrow M$ to $E \longrightarrow M'$. The deformations of 2-dimensional compact complex manifolds in the first step is calculated in Section 1. The deformation in the second step can be calculated by the method given in Section 2. We will do the calculation for the second step for the case where M is a complex two-dimensional Hermitian manifold.

Set-up: Let M be a compact Hermitian manifold of dimension 2 and V be a rank 2 Hermitian vector bundle over M. Since $H^q(M, *)$ will vanish for $q > 2$, the Euler characteristic is given by

$$\chi(M, End_0V) = h^0(M, End_0V) - h^1(M, End_0V) + h^2(M, End_0V).$$

Hence the inequality (91) is translated into

$$m_0(V) \geq h^0(M, End_0V) - \chi(M, End_0V) \geq -\chi(M, End_0V) \tag{93}$$

where $m_0(V)$ is the freedom of deformations with fixed base manifold and fixed determinant line bundle of V locally.

We will calculate $\chi(M, End_0V)$ by Riemann-Roch theorem first and substitute $V = TM$ to get $\chi(M, End_0TM)$. Suppose M is of Chern classes $c_0 = 1, c_1$ and c_2 with formal factorization $1 + c_1x + c_2x^2 = (1 + \gamma_1x)(1 + \gamma_2x)$. Suppose $End_0(V)$ is of Chern classes $d_0 = 1, d_1, d_2$ and d_3 with formal factorization

$1 + d_1x + d_2x^2 + d_3x^3 = (1 + \delta_1 x)(1 + \delta_2 x)(1 + \delta_3 x)$, then

$$\begin{aligned}
\chi(M, End_0V) &= \varkappa_2 \left[e^{c_1/2} \frac{\gamma_1/2}{\sinh \gamma_1/2} \frac{\gamma_2/2}{\sinh \gamma_2/2} (e^{\delta_1} + e^{\delta_2} + e^{\delta_3}) \right] \\
&= \varkappa_2[(1 + \frac{1}{2}c_1 + \frac{1}{8}c_1^2)(1 - \frac{1}{24}\gamma_1^2)(1 - \frac{1}{24}\gamma_2^2) \\
&\quad (1 + \delta_1 + \frac{1}{2}\delta_1^2 + 1 + \delta_2 + \frac{1}{2}\delta_2^2 + 1 + \delta_3 + \frac{1}{2}\delta_1^3)] \\
&= \varkappa_2 \left[(1 + \frac{1}{2}c_1 + \frac{1}{8}c_1^2)(1 - \frac{1}{24}c_1^2 + \frac{1}{12}c_2)(3 + d_1 + \frac{1}{2}d_1^2 - d_2) \right] \\
&= \varkappa_2 \left[(1 - \frac{1}{24}c_1^2 + \frac{1}{12}c_2 + \frac{1}{2}c_1 + \frac{1}{8}c_1^2)(3 + d_1 + \frac{1}{2}d_1^2 - d_2) \right] \\
&= \varkappa_2 \left[(1 + \frac{1}{2}c_1 + \frac{1}{12}c_1^2 + \frac{1}{12}c_2)(3 + d_1 + \frac{1}{2}d_1^2 - d_2) \right] \\
&= \frac{1}{2}d_1^2 - d_2 + \frac{1}{2}c_1d_1 + \frac{1}{4}c_1^2 + \frac{1}{4}c_2
\end{aligned}$$

Before calculating d_1 and d_2 for End_0V, we state the property of Chern classes form [Hir].

Proposition 3.1. *Let ξ be a $U(q)$-bundle and ξ' a $U(q')$-bundle over an admissible space M, consider factorizations*

$$\sum_{i=0}^{q} c_i(\xi)x^i = \Pi_{j=1}^{q}(1 + \gamma_j x), \quad \sum_{i=0}^{q'} c_i(\xi')x^i = \Pi_{k=1}^{q'}(1 + \delta_k x).$$

Then,

I) $\sum_{i=0}^{q} c_i(\xi^*)x^i = \Pi_{j=1}^{q}(1 - \gamma_j x)$, *i.e.* $c_i(\xi^*) = (-1)^i c_i(\xi)$.
II) $\sum_{i=0}^{q+q'} c_i(\xi \oplus \xi')x^i = \Pi_{j=1}^{q}(1+\gamma_j x)\Pi_{k=1}^{q'}(1+\delta_k x)$, *i.e.* $c(\xi \oplus \xi') = c(\xi)c(\xi')$.
III) $\sum_{i=0}^{qq'} c_i(\xi \otimes \xi')x^i = \Pi_{j,k}(1 + (\gamma_j + \delta_k)x), \quad 1 \le j \le q, 1 \le k \le q'$.

Suppose V is of Chern classes $a_0 = 1, a_1$ and a_2 with $1 + a_1x + a_2x^2 = (1+\alpha_1 x)(1+\alpha_2 x)$ then by the above theorem V^* is of Chern classes $a_0 = 1, -a_1$ and a_2 with $1 - a_1x + a_2x^2 = (1 + \beta_1 x)(1 + \beta_2 x)$. Since

$$V^* \otimes V = End(V) = End_0V \oplus L$$

where L is a line bundle with Chern class $f_1 = 0$ and $V^* \otimes V$ is of Chern classes $e_0 = 1, e_1, e_2, e_3$ and e_4, then

$$\sum_{i=0}^{2\times 2} e_i(V^* \otimes V)x^i = \sum_{i=0}^{2+2} e_i(End_0V \oplus L)x^i.$$

However, we have

$$
\begin{aligned}
& \sum_{i=0}^{2\times 2} e_i(V^*\otimes V)x^i \\
= \;& (1+(\beta_1+\alpha_1)x)(1+(\beta_1+\alpha_2)x)(1+(\beta_2+\alpha_1)x)(1+(\beta_2+\alpha_2)x) \\
= \;& (1+(\beta_1+\alpha_1+\beta_1+\alpha_2)x+(\beta_1+\alpha_1)(\beta_1+\alpha_2)x^2) \\
& (1+(\beta_2+\alpha_1+\beta_2+\alpha_2)x+(\beta_2+\alpha_1)(\beta_2+\alpha_2)x^2+e_3x^3+e_4x^4 \\
= \;& (1+(2\beta_1+a_1)x+(\beta_1^2+\beta_1a_1+a_2)x^2) \\
& (1+(2\beta_2+a_1)x+(\beta_2^2+\beta_2a_1+a_2)x^2) \\
= \;& 1+(2\beta_2+a_1)x+(\beta_2^2+\beta_2a_1+a_2)x^2+(2\beta_1+a_1)x \\
& +(2\beta_1+a_1)(2\beta_2+a_1)x^2+(\beta_1^2+\beta_1a_1+a_2)x^2+e_3x^3+e_4x^4 \\
= \;& 1+0x+(4a_2-a_1^2)x^2+e_3x^3+e_4x^4
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{i=0}^{2+2} e_i(End_0V\oplus L)x^i \\
= \;& (1+\delta_1x)(1+\delta_2x)(1+\delta_3x)(1+f_1x) \\
= \;& 1+d_1x+d_2x^2+d_3x^3.
\end{aligned}
$$

By comparing, we get $d_1=0$ and $d_2=4a_2-a_1^2$. Therefore,

$$\chi(M,End_0V)=\frac{1}{2}d_1^2-d_2+\frac{1}{2}c_1d_1+\frac{1}{4}c_1^2+\frac{1}{4}c_2=-4a_2+a_1^2+\frac{1}{4}c_1^2+\frac{1}{4}c_2.$$

In particular when $V=TM$ we have $a_i=c_i, i=0,1,2$. Then

$$\chi(M,End_0TM)=-4c_2+c_1^2+\frac{1}{4}c_1^2+\frac{1}{4}c_2=\frac{5}{4}(c_1^2-3c_2).$$

There is a computation of this number in [C-W-Y]. By the inequality (93) when $V=End_0(TM)$, we have the bound of freedom of deformations with fixed base manifold and fixed determinant of TM locally as

$$m_0(TM)\geq -\frac{5}{4}(c_1^2-3c_2).$$

Hence when $c_1^2\geq 3c_2$, we have no conclusion about TM's fixed base manifold and fixed determinant deformations locally. In particular, in Chapter 3, we have that if M is a compact Kähler surface with strictly negative sectional curvature and $c_1^2=3c_2$, then M is biholomorphic to a ball with Bergman metric. There is an open problem asking whether a ball has deformation itself. On the other hand for any surface satisfying $c_1^2<3c_2$, in particular the compact Kähler surfaces which do not satisfies $c_1^2=3c_2$, TM will have fixed base manifold and fixed determinant deformations since $m_0(TM)>0$ in that case.